# Reflections from Circular Bends in Rectangular Wave Guides—Matrix Theory

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A method of computing reflections produced by circular bends in rectangular wave guides is presented. The procedure employs the theory of matrices. Although the matrix equations are quite simple, a considerable amount of calculation is necessary before quantitative results may be obtained. Fortunately, the approximate formulas pertaining to gentle bends hold surprisingly well for rather sharp bends. These formulas are obtained by a limiting process from the matrix equations. The approximate formula for reflection from an H-bend (in which the magnetic vector lies in the plane of the bend) generalizes an earlier result due to R. E. Marshak. The corresponding formula for the E-bend appears to be new.

## Introduction

A NUMBER of investigators have studied the propagation of electromagnetic waves in a bent pipe of rectangular cross-section, the bend being along an arc of a circle. H. Buchholz[1], S. Morimoto[2], and W. J. Albersheim[3] have employed Bessel functions to express the field in the bend. The form assumed by the field when the radius of curvature of the bend becomes large has been obtained by K. Riess[4] and R. E. Marshak[5] who use approximations suited to this case. Marshak also obtains expressions for various reflection and transmission coefficients. A discussion of the subject using rather simple but approximate analysis is given on pages 324–330 of a text book[6] by S. A. Schelkunoff. The Bessel function approach is also sketched in the same section.

Here we study the disturbance produced when a wave goes around a circular bend (of some given angle) in a rectangular wave guide, the guide being straight on either side of the bend. Especial attention is paid to the dominant mode reflection coefficients $g_{\overline{10}}$ and $d_{\overline{01}}$ corresponding to H-bends and E-bends, respectively. As equations (4.2–6) and (4.4–4) show, these reflection coefficients (which are of the nature of voltage rather than power reflection coefficients) vary inversely as the square of the radius of curvature of the bend when the bend is gentle. The substance of (4.2–6) has been given by Marshak[5] for the important case in which only the dominant mode is propagated and the angle of the bend not too small.

When the bend is so sharp that the formulas mentioned above do not apply the reflection coefficients may be computed from the rather simple looking matrix expressions (2.3–3) together with (2.3–4). However, their appearance is deceptive and, as is shown by the numerical work in Part V, considerable labor is necessary to obtain an answer.

The gentle bend formulas were obtained from the matrix equations by the limiting process described in Part III. It seems likely that the matrix method, which is similar to the method used in an earlier paper[7] on transmission line equations, may be applied to other wave guide problems. With this thought in mind, the development of Parts II and III has been couched in general terms.

The matrices used in the present theory are of infinite order since the guide may support an infinite number of modes of propagation. This fact makes it difficult to justify all the steps in our analysis, and we do not attempt to do so.* Despite this lack of rigor, I believe that the procedures given here lead to the correct results since they yield, for gentle bends, expressions obtained by Buchholz and Marshak. Moreover, although numerical results tabulated in Part V were obtained by using matrices of only the second and third order, they indicate a rapid convergence as the matrix order is increased.

## PART I

## PROPAGATION OF WAVES IN GUIDE

### 1.1 *Propagation in a Straight Wave Guide*

Rather general expressions for the electric and magnetic intensities $E$ and $H$ in a field are (see pp. 127–128 of Reference[6])

$$\begin{aligned} E &= -i\omega\mu\vec{A} + \frac{1}{i\omega\epsilon}\text{ grad div }\vec{A} - \text{curl }\vec{B} \\ H &= \text{curl }\vec{A} + \frac{1}{i\omega\mu}\text{ grad div }\vec{B} - i\omega\epsilon\vec{B} \end{aligned} \tag{1.1–1}$$

The field is assumed to vary with the time $t$ as $e^{i\omega t}$, $\omega$ is the radian frequency, $\mu$ the permeability and $\epsilon$ the dielectric constant (for free space $\mu = 1.257 \times 10^{-6}$ henries/meter, $\epsilon = 8.854 \times 10^{-12}$ farads/meter). The vector potentials $\vec{A}$ and $\vec{B}$ satisfy the wave equations

$$\begin{aligned} \nabla^2\vec{A} = \sigma^2\vec{A}, &\qquad \nabla^2\vec{B} = \sigma^2\vec{B} \\ \nabla^2 &\equiv \text{Laplacian operator} \\ \sigma^2 &= \omega^2\mu\epsilon \end{aligned} \tag{1.1–2}$$

In dealing with bends, it is convenient to choose $\vec{A}$ and $\vec{B}$ normal to the plane of the bend. In our notation, this plane is always taken to be the $x$, $z$ plane so that $\vec{A}$ and $\vec{B}$ are parallel to the $y$ axis. The $z$ axis is parallel

* Similar questions arise in the rigorous treatment of an infinite set of linear equations. A discussion of this subject is given in Chap. III of Reference[8].

to the guide axis and, for the straight guide of the section, the guide walls are sections of the planes $x = 0, x = a, y = 0, y = b$.

Thus, a general wave traveling in the positive $z$ direction may be described by the two functions (which represent the magnitudes of $\vec{A}$ and $\vec{B}$)

$$A = \sum_{m,n} g_{mn}^{+} e^{-\Gamma_{mn}z} \sin(\pi mx/a)\cos(\pi ny/b) \tag{1.1–3}$$

$$m = 1, 2, 3, \cdots; \qquad n = 0, 1, 2, \cdots$$

$$B = \sum_{m,n} d_{mn}^{+} e^{-\Gamma_{mn}z} \cos(\pi mx/a)\sin(\pi ny/b) \tag{1.1–4}$$

$$m = 0, 1, 2, \cdots; \qquad n = 1, 2, 3, \cdots$$

where the coefficients $g_{mn}^{+}$ and $d_{mn}^{+}$ are constants and the plus signs indicate propagation in the positive $z$ direction.

The propagation constant $\Gamma_{mn}$ is obtained from

$$\Gamma_{mn}^2 = \sigma^2 + (\pi m/a)^2 + (\pi n/b)^2, \sigma = i2\pi/\lambda_0, \tag{1.1–5}$$

$$\lambda_0 = \text{wavelength in free space.}$$

Equation (1.1–5) arises when the typical term in (1.1–3) is substituted for $A$ in the equation

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2} = \sigma^2 A \tag{1.1–6}$$

This and a similar equation for $B$ are the forms assumed by (1.1–2) for the rectangular coordinates of our straight guide.

The electric and magnetic intensities in the guide are given by

$$\begin{aligned}
E_x &= \frac{1}{i\omega\epsilon}\frac{\partial^2 A}{\partial x \partial y} + \frac{\partial B}{\partial z} & H_x &= -\frac{\partial A}{\partial z} + \frac{1}{i\omega\mu}\frac{\partial^2 B}{\partial x \partial y} \\
E_y &= -i\omega\mu A + \frac{1}{i\omega\epsilon}\frac{\partial^2 A}{\partial y^2} & H_y &= -i\omega\epsilon B + \frac{1}{i\omega\mu}\frac{\partial^2 B}{\partial y^2} \\
E_z &= \frac{1}{i\omega\epsilon}\frac{\partial^2 A}{\partial z \partial y} - \frac{\partial B}{\partial x} & H_z &= \frac{\partial A}{\partial x} + \frac{1}{i\omega\mu}\frac{\partial^2 B}{\partial z \partial y}
\end{aligned} \tag{1.1–7}$$

which follow from (1.1–1).

It is seen that the wave is completely specified by the $g_{mn}^{+}$'s and $d_{mn}^{+}$'s. These may be arranged as (infinite) column matrices in any convenient order. Thus in dealing with (1.1–3) and (1.1–4) we may write

$$g^{+} = \begin{bmatrix} g_{10}^{+} \\ g_{20}^{+} \\ g_{11}^{+} \\ g_{30}^{+} \\ g_{21}^{+} \\ g_{12}^{+} \end{bmatrix} \qquad d^{+} = \begin{bmatrix} d_{01}^{+} \\ d_{02}^{+} \\ d_{11}^{+} \\ d_{03}^{+} \\ \cdot \\ \cdot \end{bmatrix} \tag{1.1–8}$$

In our work we shall consider only those modes corresponding to a fixed value of $m$ (or of $n$) and the order is almost automatically fixed.

The factors which determine the propagation of the typical terms in the summations (1.1–3) and (1.1–4) for $A$ and $B$ are

$$\alpha_{mn}(z) = g_{mn}^{+} e^{-z\Gamma_{mn}}, \qquad \beta_{mn}(z) = d_{mn}^{+} e^{-z\Gamma_{mn}} \tag{1.1–9}$$

The column matrices obtained by arranging these quantities in the same order as in (1.1–8) will be denoted by $\alpha(z)$ and $\beta(z)$. We may write

$$\alpha(z) = e^{-z\Gamma_\alpha} g^+, \qquad \beta(z) = e^{-z\Gamma_\beta} d^+ \tag{1.1–10}$$

where $\exp(-z\Gamma_\alpha)$ and $\exp(-z\Gamma_\beta)$ are square matrices defined by power series each term of which is a square matrix:

$$e^{-z\Gamma} = I - \frac{z\Gamma}{1!} + \frac{z^2\Gamma^2}{2!} - \frac{z^3\Gamma^3}{3!} + \cdots \tag{1.1–11}$$

$I$ is the unit matrix and $\Gamma_\alpha$ is the diagonal matrix*

$$\Gamma_\alpha = \begin{bmatrix} \Gamma_{10} & 0 & 0 & \cdot \\ 0 & \Gamma_{20} & 0 & \cdot \\ 0 & 0 & \Gamma_{11} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{1.1–12}$$

in which the order of the diagonal elements is the same as the order of the elements in the column matrix $g^+$. Similarly $\Gamma_\beta$ is a diagonal matrix whose elements are $\Gamma_{01}, \Gamma_{02}, \Gamma_{11}, \Gamma_{03}, \cdots$, the order being fixed by $d^+$. When $\Gamma$ is replaced by $\Gamma_\alpha$ in (1.1–11) it is easy to obtain $\Gamma_\alpha^2$, $\Gamma_\alpha^3$, etc. and sum the resulting series to obtain

$$e^{-z\Gamma_\alpha} = \begin{bmatrix} e^{-z\Gamma_{10}} & 0 & 0 & \cdot \\ 0 & e^{-z\Gamma_{20}} & 0 & \cdot \\ 0 & 0 & e^{-z\Gamma_{11}} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{1.1–13}$$

A similar expression exists for $\exp(-z\Gamma_\beta)$. The expression (1.1–10) for $\alpha(z)$ is seen to be true when the square matrix (1.1–13) is multiplied, by matrix multiplication, into the column $g^+$.

It turns out that the field in a circular bend (in a rectangular guide) may be represented by a generalization of the foregoing expressions. In this generalization, which will be studied in the following sections, the square matrices $\Gamma_\alpha$ and $\Gamma_\beta$ no longer have the simple form of diagonal matrices.

* That is, a square matrix in which all of the elements other than those in the principal diagonal are zero.

### 1.2 *Propagation in a Circular Bend*

In dealing with a circular bend we choose cylindrical coordinates $(\rho, \varphi, y)$ as shown in Fig. 1. With these coordinates we associate new coordinates, shown in Figs. 1 and 2, $(x, y, z)$ which have approximately the same significance as in the straight guide. $z$ is the distance measured along the axis of

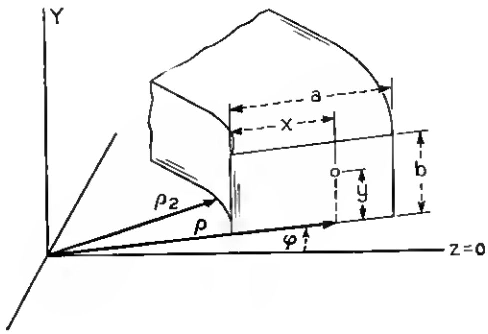


Fig. 1

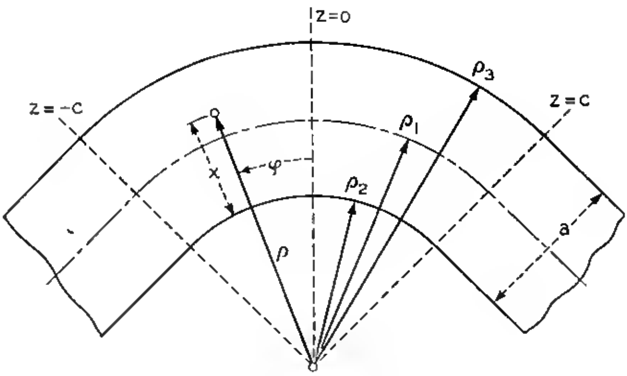


Fig. 2

the guide (defined as the locus of the centers of gravity of the transverse cross-sections of the guide), and $x$ and $y$ are the transverse coordinates.

Let $\rho = \rho_1 = (\rho_2 + \rho_3)/2 = \rho_2 + a/2$ be the radius of curvature of the guide axis, and let the origin of the polar coordinates be taken at the center of curvature. Then $z$ is equal to $-\rho_1\varphi$ where the minus sign is necessary to make $(x, y, z)$ a right-handed coordinate system. Since the vertical (in

Fig. 1) walls are to be specified by $x = 0$ and $x = a$ we set $x = \rho - \rho_1 + a/2$. Thus, the two sets of coordinates are related by

$$\begin{aligned} \rho &= x + \rho_1 - a/2 = x + \rho_2 \\ \varphi &= -z/\rho_1 \\ y &= y \end{aligned} \tag{1.2–1}$$

where $\rho_1$, $\rho_2$ and $a$ are constants.

We again choose $\vec{A}$ and $\vec{B}$ in (1.1–1) to be parallel to the $y$ axis. In the cylindrical coordinates,

$$\begin{aligned} E_\rho &= \frac{1}{i\omega\epsilon}\frac{\partial^2 A}{\partial\rho\partial y} - \frac{1}{\rho}\frac{\partial B}{\partial\varphi} & H_\rho &= \frac{1}{\rho}\frac{\partial A}{\partial\varphi} + \frac{1}{i\omega\mu}\frac{\partial^2 B}{\partial\rho\partial y} \\ E_\varphi &= \frac{1}{i\omega\epsilon\rho}\frac{\partial^2 A}{\partial\varphi\partial y} + \frac{\partial B}{\partial\rho} & H_\varphi &= -\frac{\partial A}{\partial\rho} + \frac{1}{i\omega\mu\rho}\frac{\partial^2 B}{\partial\varphi\partial y} \\ E_y &= -i\omega\mu A + \frac{1}{i\omega\epsilon}\frac{\partial^2 A}{\partial y^2} & H_y &= -i\omega\epsilon B + \frac{1}{i\omega\mu}\frac{\partial^2 B}{\partial y^2} \end{aligned} \tag{1.2–2}$$

where now, from (1.1–2), $A$ satisfies the wave equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial A}{\partial\rho}\right] + \frac{1}{\rho^2}\frac{\partial^2 A}{\partial\varphi^2} + \frac{\partial^2 A}{\partial y^2} = \sigma^2 A \tag{1.2–3}$$

and likewise for $B$.

One method of dealing with (1.2–3) which is sometimes used is to assume

$$A = e^{ip\varphi} \times (\text{sine or cosine function of } y) \times f(\rho) \tag{1.2–4}$$

where $f(\rho)$ turns out to be a Bessel function of order $p$ with its argument proportional to $\rho$. However, we shall proceed in a different direction.

The change of coordinates (1.2–1) transforms (1.2–2) into

$$\begin{aligned} E_x &= E_\rho = \frac{1}{i\omega\epsilon}\frac{\partial^2 A}{\partial x\partial y} + \frac{\rho_1}{\rho}\frac{\partial B}{\partial z} & H_x &= H_\rho = -\frac{\rho_1}{\rho}\frac{\partial A}{\partial z} + \frac{1}{i\omega\mu}\frac{\partial^2 B}{\partial x\partial y} \\ E_y &= -i\omega\mu A + \frac{1}{i\omega\epsilon}\frac{\partial^2 A}{\partial y^2} & H_y &= -i\omega\epsilon B + \frac{1}{i\omega\mu}\frac{\partial^2 B}{\partial y^2} \\ E_z &= -E_\varphi = \frac{\rho_1}{i\omega\epsilon\rho}\frac{\partial^2 A}{\partial z\partial y} - \frac{\partial B}{\partial x} & H_z &= -H_\varphi = \frac{\partial A}{\partial x} + \frac{\rho_1}{i\omega\mu\rho}\frac{\partial^2 B}{\partial z\partial y} \end{aligned} \tag{1.2–5}$$

and (1.2–3) into

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\rho_1^2}{\rho^2}\frac{\partial^2 A}{\partial z^2} + \frac{1}{\rho}\frac{\partial A}{\partial x} - \sigma^2 A = 0, \tag{1.2–6}$$

where $\rho_1$ is a constant and $\rho = x + \rho_1 - a/2$ is to be considered a function of $x$. To solve (1.2–6) and the corresponding equation for $B$ we assume

$$A = \sum_{m,n} \alpha_{mn}(z) \sin (\pi m x/a) \cos (\pi n y/b)$$
$$m = 1, 2, 3, \cdots ; \qquad n = 0, 1, 2, \cdots \tag{1.2–7}$$

$$B = \sum_{m,n} \beta_{mn}(z) \cos (\pi m x/a) \sin (\pi n y/b)$$
$$m = 0, 1, 2, \cdots ; \qquad n = 1, 2, 3, \cdots \tag{1.2–8}$$

these expressions being suggested by (1.1–3) and (1.1–4). The expressions (1.2–5) for the electric intensity show that this choice of $A$ and $B$ make its tangential component vanish at the walls of the guide. Thus the boundary conditions are satisfied.

In order to determine $\alpha_{mn}(z)$ so that tbe differential equation for $A$ is satisfied, we substitute (1.2–7) in (1.2–6). The resulting left hand side of (1.2–6) may be regarded as a function, say $f(x, y)$, of $x$ and $y$ with the $\alpha$'s and their derivatives entering as parameters. We must choose the $\alpha$'s so as to make this function zero. Relations which must be satisfied by the $\alpha$'s may be obtained by expanding $f(x, y)$ in a double Fourier series for which the typical term is a coefficient times $\sin (\pi m x/a) \cos (\pi n y/b)$, and then setting the coefficient of each term to zero. This form of expansion is suggested by (1.2–7). However, it should be mentioned that such an expansion is best suited to a function which vanishes at $x = 0$ and $x = a$, a condition not fulfilled by $f(x, y)$ because of the term $\rho^{-1}\partial A/\partial x$ in (1.2–6). This causes no real trouble because our region of representation runs only from $x = 0$ to $x = a$ and hence our series is no worse than the Fourier sine series for the periodic function (of period $2a$) which is $-1$ for $-a < x < 0$ and $+1$ for $0 < x < a$.

To carry out the procedure outlined above, we multiply (1.2–6) (after putting in (1.2–7)) by $\sin(\pi p x/a) \cos(\pi \ell y/b)$ and integrate $x$ from 0 to $a$ and $y$ from 0 to $b$. Using the expression (1.1–5) for $\Gamma^2_{mn}$ and reducing gives

$$-\Gamma^2_{p\ell}\, \alpha_{p\ell}(z) + \sum_{m=1}^{\infty} [P_{pm}\, \alpha''_{m\ell}(z) - S_{pm}\, \alpha_{m\ell}(z)] = 0 \tag{1.2–9}$$

where $p$ may have any one of the values $1, 2, 3, \cdots$ and the double prime on $\alpha$ denotes the second derivative with respect to $z$. The $P$'s and $S$'s are constants given by

$$P_{pm} = (2/a) \int_0^a (\rho_1^2/\rho^2) \sin (\pi p x/a) \sin (\pi m x/a)\, dx, \tag{1.2–10}$$

$$S_{pm} = -2\pi m a^{-2} \int_0^a \sin (\pi p x/a) \cos (\pi m x/a)\, dx/\rho \tag{1.2–11}$$

The evaluation of these integrals is discussed in Appendix I. Thus (1.2–9) is the $p^{\text{th}}$ equation of a set of differential equations to be solved simultaneously for $\alpha_{1\ell}(z)$, $\alpha_{2\ell}(z)$, $\cdots$.

The customary method of solving a set of equations such as (1.2–9) is to assume that all the $\alpha$'s vary as $e^{\gamma z}$ so that for each $\alpha_{m\ell}(z)$ we may write $e^{\gamma z} g_{m\ell}$. This leads to a set of simultaneous homogeneous linear equations for the constants $g_{m\ell}$. In order that these equations may have a solution the determinant of the coefficients must vanish. Since the only derivative of $\alpha_{m\ell}(z)$ contained in (1.2–9) is the second, $\gamma$ appears in the determinant only as $\gamma^2$. Let $\gamma_1^2, \gamma_2^2, \gamma_3^2, \cdots$ be the values of $\gamma^2$ which cause the determinant to vanish and let $k_{1j}, k_{2j}, \cdots$ be the values of $g_{1\ell}, g_{2\ell}, \cdots$ corresponding to $\gamma^2 = \gamma_j^2$. The $k$'s are determined to within an arbitrary multiplying constant which, for the sake of convenience, is chosen so that $k_{jj} = 1$.

Thus one solution of the differential equation (1.2–6) is

$$A = e^{-\gamma_j z} \cos(\pi \ell y/b) \sum_{m=1}^{\infty} k_{mj} \sin(\pi m x/a). \tag{1.2–13}$$

This particular solution corresponds to the $j^{\text{th}}$ one of the modes (traveling in the positive $z$ direction) for which $A$ is proportional to $\cos(\pi \ell y/b)$.

In much the same way it may be shown that the series (1.2–8) assumed for $B$ is a solution of equation (1.2–6) (with $A$ replaced by $B$) provided the coefficients $\beta_{mn}(z)$ satisfy the set of equations

$$-\Gamma_{p\ell}^2 \beta_{p\ell}(z) + \sum_{m=0}^{\infty} [Q_{pm} \beta_{m\ell}''(z) - U_{pm} \beta_{m\ell}(z)] = 0 \tag{1.2–14}$$

for $p = 0, 1, 2, \cdots$ and $\ell = 1, 2, 3, \cdots$. Here

$$Q_{pm} = (\epsilon_p/a) \int_0^a (\rho_1^2/\rho^2) \cos(\pi p x/a) \cos(\pi m x/a)\, dx \tag{1.2–15}$$

$$U_{pm} = \pi m \epsilon_p a^{-2} \int_0^a \cos(\pi p x/a) \sin(\pi m x/a)\, dx/\rho \tag{1.2–16}$$

where $\epsilon_0 = 1$ and $\epsilon_p = 2$ for $p > 0$. These integrals are discussed in Appendix I.

The problem of determining the reflection from a bend in a wave guide involves considerable manipulation of equations (1.2–9) and (1.2–14). The introduction of matrix notation in the manner suggested by the work of Section 1.1 for straight guides simplifies this work. Although $\alpha_{mn}(z)$ and $\beta_{mn}(z)$ are no longer the simple exponential functions given by (1.1–9), it turns out that the column matrices $\alpha(z)$ and $\beta(z)$ are still given by (for a wave traveling in the positive $z$ direction) by the matrix expression (1.1–10). As mentioned earlier, $\Gamma_\alpha$ and $\Gamma_\beta$ are no longer simple diagonal matrices.

We now turn to the task of expressing (1.2–9) and (1.2–14) in matrix form.

1.3 *Propagation Constant Matrix for Curved Rectangular Guide*

From this point onward in our investigation of propagation in the rectangular guide we shall assume $A$ to be proportional to $\cos(\pi \ell y/b)$. Thus instead of the general expression (1.2–7) for $A$ we shall deal with the more restricted form

$$A = \cos(\pi \ell y/b) \sum_{m=1}^{\infty} \alpha_{m\ell}(z) \sin(\pi m x/a) \tag{1.3–1}$$

where $\ell$ has one of the values 0, 1, 2, 3, $\cdots$. Since the most general disturbance may be obtained by the superposition of disturbances of the form (1.3–1) no real generality will be lost.

The introduction of (1.3–1) is suggested by the fact that the set $\alpha_{1\ell}(z)$, $\alpha_{2\ell}(z)$, $\cdots$ may be determined from (1.2–9) (at least to within arbitrary constants of integration) without considering the other $\alpha_{mn}(z)$'s, $n \neq \ell$. The introduction of (1.3–1) is also suggested by physical reasons. The plane of the bend is the $z$, $x$ plane and there is nothing in the system tending to change the field distribution in the $y$ direction.

Equation (1.2–13) is a special case of (1.3–1). Furthermore the most general form of (1.3–1) (corresponding to a wave progressing in the positive $z$ direction) may be obtained by multiplying (1.2–13) by an arbitrary constant $c_j$ and summing on $j$.

In order to write the set of differential equations (1.2–9) for the $\alpha_{m\ell}(z)$'s in matrix form we introduce the infinite matrices

$$\Gamma_0 = \begin{bmatrix} \Gamma_{1\ell} & 0 & 0 & \cdot \\ 0 & \Gamma_{2\ell} & 0 & \cdot \\ 0 & 0 & \Gamma_{3\ell} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \qquad \alpha(z) = \begin{bmatrix} \alpha_{1\ell}(z) \\ \alpha_{2\ell}(z) \\ \alpha_{3\ell}(z) \\ \cdot \end{bmatrix}$$

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdot \\ P_{21} & P_{22} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}, \qquad S = \begin{bmatrix} S_{11} & S_{12} & \cdot \\ S_{21} & S_{22} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \tag{1.3–2}$$

where the elements of $\Gamma_0$ are obtained by setting $n = \ell$ in equation (1.1–5) for $\Gamma_{mn}$, and the elements of $P$ and $Q$ are given by the integrals (1.2–10) and (1.2–11). The rules of matrix multiplication then show that (1.2–9) is the $p^{\text{th}}$ element of the matrix equation

$$P\alpha''(z) - (\Gamma_0^2 + S)\alpha(z) = 0 \tag{1.3–3}$$

Premultiplying by $P^{-1}$ ccnverts this equation into

$$\alpha''(z) - \Gamma_\alpha^2\, \alpha(z) = 0 \tag{1.3–4}$$

where

$$\Gamma_\alpha^2 = P^{-1}(\Gamma_0^2 + S) \tag{1.3–5}$$

It may be verified by direct differentiation of the series (1.1–11) defining $\exp(-z\Gamma)$ that a solution of (1.3–4) is

$$\alpha(z) = e^{-z\Gamma_\alpha} g^+ \tag{1.3–6}$$

where, as in (1.1–10) for the straight guide, $g^+$ is a column of constants (of integration). However, now $\Gamma_\alpha$ is to be obtained by taking the square root of tbe right hand side of (1.3–5), a process which is not easy since it usually requires one to obtain the characteristic roots and modal columns of $\Gamma_\alpha^2$ (see equation (1.3–10)).

As far as (1.3–6) being a solution of the differential equation is concerned, $\Gamma_\alpha$ may be any matrix whose square is given by (1.3–5). We shall restrict it as follows: When $\xi = a/\rho_1$ becomes small, as in the case of a gentle bend, it is seen from (1.2–10) and (1.2–11) that $P$ approaches the unit matrix and $S$ approaches zero. Hence, $\Gamma_\alpha^2$ approaches the diagonal matrix $\Gamma_0^2$. $\Gamma_\alpha$ is chosen so that it approaches $\Gamma_0$, that is, all of the elements in the principal diagonal are either positive real or positive imaginary. This makes $\exp(-z\Gamma_\alpha)$ approach the diagonal matrix $\exp(-z\Gamma_0)$. With this choice of $\Gamma_\alpha$ the expression (1.3–6) for $\alpha(z)$ corresponds to a wave traveling in the positive $z$ direction.

The various modes of propagation in the bend may be obtained from $\Gamma_\alpha^2$ by expressing, in matrix notation, the steps leading to (1.2–13) (which gives $A$ for the $j^{\text{th}}$ mode). We assume $\alpha(z)$ to be the column matrix obtained by multiplying the column matrix $g$ of constants by the scalar quantity $e^{\gamma z}$. Setting this in (1.3–4) gives

$$(\gamma^2 I - \Gamma_\alpha^2)g = 0 \tag{1.3–7}$$

where $I$ is the unit matrix. In order that (1.3–7) may have a solution, the determinant of the coefficient of $g$ must vanish. This leads to the characteristic equation* for $\gamma^2$:

$$|\gamma^2 I - \Gamma_\alpha^2| = 0 \tag{1.3–8}$$

The vertical bars denote the determinant of the inclosed matrix. The roots $\gamma_1^2, \gamma_2^2, \cdots$ are therefore the latent (or characteristic) roots of $\Gamma_\alpha^2$. If we let $k_j$ denote** the column $g$ obtained when $\gamma = \gamma_j$ in (1.3–7) then

* See Section 3.6 of Reference[9].

** We choose this notation in order to adhere as closely as possible to that of Reference[9]. Incidentally, the column $k_j$ is proportional to the $j^{\text{th}}$ column of $\kappa^{-1}$ where $\kappa$ is the modal row matrix introduced in Section 5.1.

$$(\gamma_j^2 I - \Gamma_\alpha^2)\, k_j = 0 \tag{1.3–9}$$

and the elements $k_{1j}$, $k_{2j}$, $\cdots$ of the modal column $k_j$ are the ones appearing as coefficients in (1.2–13).

Equation (1.3–9) and the methods of matrix analysis lead to

$$\Gamma_\alpha^2 = k[\gamma^2]_d\, k^{-1}, \qquad \Gamma_\alpha = k\,[\gamma]_d\, k^{-1} \tag{1.3–10}$$

where $k$ is the square matrix whose $j^{\text{th}}$ column is $k_j$ and $[\gamma^2]_d$, $[\gamma]_d$ are diagonal matrices having $\gamma_j^2, \gamma_j$ as the $j^{\text{th}}$ elements in their principal diagonals. The representation (1.3–10) certainly holds for the rectangular guide since in this case no repeated roots occur.

In analogy with the expression (1.3–1) for $A$ we shall henceforth deal with $B$ in the form

$$B = \sin(\pi \ell y/b) \sum_{m=0}^{\infty} \beta_{m\ell}(z) \cos(\pi m x/a) \tag{1.3–11}$$

where $\ell$ has one of the values $1, 2, 3, \cdots$. In much the same way as before it may be shown that for a wave traveling along the bend in the positive direction the $\beta_{m\ell}(z)$'s in (1.3–11) are given by

$$\beta(z) = e^{-z\Gamma_\beta}\, d^+ \tag{1.3–12}$$

where $d^+$ is a column of arbitrary constants and

$$\Gamma_\beta^2 = Q^{-1}(\Gamma_0^2 + U) \tag{1.3–13}$$

In (1.3–12) and (1.3–13)

$$\Gamma_0 = \begin{bmatrix} \Gamma_{0\ell} & 0 & 0 & \cdot \\ 0 & \Gamma_{1\ell} & 0 & \cdot \\ 0 & 0 & \Gamma_{2\ell} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad \beta(z) = \begin{bmatrix} \beta_{0\ell}(z) \\ \beta_{1\ell}(z) \\ \beta_{2\ell}(z) \\ \cdot \end{bmatrix}$$

$$Q = \begin{bmatrix} Q_{00} & Q_{01} & \cdot \\ Q_{10} & Q_{11} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \qquad U = \begin{bmatrix} 0 & U_{01} & U_{02} & \cdot \\ 0 & U_{11} & U_{12} & \cdot \\ 0 & U_{21} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{1.3–14}$$

where the elements of $\Gamma_0$, $Q$ and $U$ are given by equations (1.1–5), (1.2–15) and (1.2–16), respectively.

## 1.4 *Continuity Conditions at Junction of Straight and Curved Rectangular Guides*

Electromagnetic theory requires that $E_x$, $E_y$, $H_x$ and $H_y$ be continuous in crossing a plane $z$ = constant which marks the junction of a straight and a

curved wave guide (of the same cross-section). Comparison of the first equation in (1.1–7) with the first equation in (1.2–5) shows that $E_x$ is continuous if (1)$A$ is continuous and (2) if $\partial B/\partial z$ in the straight portion is equal (the equality being taken at the junction) to $(\rho_1/\rho)\ \partial B/\partial z$ in the curved portion. Examination of the expressions for the remaining field components shows that all the continuity conditions are satisfied if, at the junction,

$$
\begin{gathered}
[A \text{ in straight portion}] = [A \text{ in bend}] \\
\left[\frac{\partial A}{\partial z} \quad `` \quad `` \quad `` \right] = \left[\frac{\rho_1}{\rho}\frac{\partial A}{\partial z} \text{ in bend}\right]
\end{gathered}
\tag{1.4–1}
$$

and likewise for $B$.

Let $A$ in the bend be given by (1.3–1) and let $\alpha(z)$ denote the column matrix of coefficients shown in (1.3–2). $A$ in the straight portion may be represented in the same way except that $\alpha(z)$ has a simpler form as explained in Section 1.1. When these expressions for $A$ are inserted in (1.4–1), both sides multiplied by $(2/a)\sin(\pi px/a)$ after cancelling out the cos $(\pi \ell y/b)$, and the results integrated with respect to $x$ from 0 to $a$ we obtain relations which may be expressed as the matrix equations

$$
\begin{gathered}
[\alpha(z) \text{ in straight portion}] = [\alpha(z) \text{ in bend}] \\
\left[\frac{d\alpha(z)}{dz} \quad `` \quad `` \quad `` \right] = \left[V\frac{d\alpha(z)}{dz} \text{ in bend}\right]
\end{gathered}
\tag{1.4–2}
$$

where $V$ is the square matrix whose $p^{\text{th}}$ row and $m^{\text{th}}$ column $(p, m = 1, 2, 3, \cdots)$ is

$$
V_{pm} = (2\rho_1/a) \int_0^a \sin(\pi px/a) \sin(\pi mx/a)\, dx/\rho, \tag{1.4–3}
$$

$\rho$ being equal to $\rho_1 + x - a/2$.

By using expression (1.3–11) for $B$ in the continuity conditions, it may be shown in much the same way that the column matrix $\beta(z)$ given by (1.3–14) must satisfy the relations

$$
\begin{gathered}
[\beta(z) \text{ in straight portion}] = [\beta(z) \text{ in bend}] \\
\left[\frac{d\beta(z)}{dz} \quad `` \quad `` \quad `` \right] = \left[W\frac{d\beta(z)}{dz} \text{ in bend}\right]
\end{gathered}
\tag{1.4–4}
$$

where $W$ is the infinite square matrix

$$
W = \begin{bmatrix} W_{00} & W_{01} & \cdot \\ W_{10} & W_{11} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \tag{1.4–5}
$$

whose elemenst are

$$W_{pm} = (\epsilon_p \rho_1/a) \int_0^a \cos(\pi px/a) \cos(\pi mx/a) \, dx/\rho \tag{1.4–6}$$
$$\epsilon_0 = 1, \qquad \epsilon_p = 2 \quad \text{for} \quad p > 0$$

Both $V_{pm}$ and $W_{pm}$ are discussed in Appendix I.

# PART II

# THEORY FOR A GENERAL WAVE GUIDE

## 2.1 *Matrix Propagation Constant for a Curved Wave Guide of Arbitrary Cross-Section*

In Section 1.3 it has heen shown that for a curved rectangular wave guide there exists a square matrix $\Gamma_\alpha$ (or $\Gamma_\beta$) which plays the same role in the propagation of a wave consisting of many modes as does the propagation constant in a simple transmission line. There $\Gamma_\alpha$ was ohtained from a special form of the wave equation which is suited to bends in rectangular guides. Here we adopt a different approach with the idea of showing that a matrix propagation constant $\Gamma$ exists under more general conditions.

The general theory of wave propagation in tubes shows that a wave traveling in the positive $z$ direction may often he represented as

$$\Phi = \sum_{j=1}^{\infty} c_j e^{-z\gamma_j} \varphi_j(x, y) \tag{2.1–1}$$

where $\Phi$ is some quantity associated with the field and is analogous to the functions $A$ and $B$ of Part I. In (2.1–1) $x$ and $y$ are transverse coordinates and $z$ a longitudinal coordinate. $\gamma_j$ is the propagation constant for the $j^{\text{th}}$ mode and $\varphi_j(x, y)$ the corresponding eigenfunction. For a circular bend in a rectangular wave guide $\varphi_j(x, y)$ is a combination of trigonometric and Bessel functions and $\gamma_j$ is proportional to the order of the Bessel functions.

We assume that we may find a set of functions $\theta_m(x, y)$, $m = 1, 2, 3, \ldots$ such that every $\varphi_j(x, y)$ may be represented as

$$\varphi_j(x, y) = \sum_{m=1}^{\infty} k_{mj} \theta_m(x, y) \tag{2.1–2}$$

The usefulness of this procedure depends upon our ahility to pick a system of $\theta_m(x, y)$'s which is appreciably simpler than the system of $\varphi_j(x, y)$'s. In the work of Part I $\theta_m(x, y)$ was taken to be the eigenfunction of the typical mode of propagation in a straight guide, i.e. the product of a sine and a cosine.

We assume further in (2.1–2) that the square matrix $k^{-1}$ exists where $k_{mj}$ is

the element in the $m^{\text{th}}$ row and $j^{\text{th}}$ column of $k$; i.e. if a root $\gamma_j$ is repeated, say, $s$ times there are $s$ linearly independent columns ($k_j$'s) corresponding to $\gamma_j$. Substitution of (2.1–2) in (2.1–1) gives

$$\begin{aligned}\Phi &= \sum_{m=1}^{\infty} \theta_m(x, y) \sum_{j=1}^{\infty} k_{mj} c_j e^{-z\gamma_j} \\ &= \sum_{m=1}^{\infty} \mu_m(z)\theta_m(x, y)\end{aligned} \tag{2.1–3}$$

where

$$\mu_m(z) = \sum_{j=1}^{\infty} k_{mj} c_j e^{-z\gamma_j} \tag{2.1–4}$$

Since $\theta_m(x, y)$ is analogous to the product of the trigonometrical terms in (1.3–1) or (1.3–11) these equations show that $\mu_m(z)$ plays the same role as $\alpha_{m\ell}(z)$ or $\beta_{m\ell}(z)$. Therefore, in accordance with the discussion given at the beginning of this section, we wish to show that the column matrix $\mu(z)$ (which is similar to $\alpha(z)$ or $\beta(z)$) whose $m^{\text{th}}$ element is $\mu_m(z)$ may be expressed as

$$\mu(z) = e^{-z\Gamma} f^+ \tag{2.1–5}$$

In this equation $\Gamma$ is a square matrix to be determined and $f^+$ is a column matrix of constants similar to $g^+$ or $d^+$.

The rules of matrix multiplication and equation (2.1–4) show that

$$\mu(z) = k[e^{-z\gamma}]_d\, c \tag{2.1–6}$$

in which $[\exp(-z\gamma)]_d$ is a diagonal matrix having $\exp(-z\gamma_j)$ as the $j^{\text{th}}$ element in its principal diagonal and $c$ is the column matrix formed from the $c_j$'s. We introduce the column $f^+$ by defining it as $\mu(0)$ whence

$$f^+ = kc, \qquad c = k^{-1} f^+ \tag{2.1–7}$$

Incidentally, from (2.1–3), the value of $\Phi$ at $z = 0$ is

$$\Phi_{z=0} = \sum_{m=1}^{\infty} f_m^+ \theta_m(x, y) \tag{2.1–8}$$

where $f_m^+$ is the $m^{\text{th}}$ element in $f^+$.

From (2.1–6) and (2.1–7)

$$\mu(z) = k[e^{-z\gamma}]_d\, k^{-1} f^+ \tag{2.1–9}$$

In this equation $k\,[\exp(-z\gamma)]_d\, k^{-1}$ is a square matrix which may be expressed as

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-z)^n}{n!} k[\gamma]_d^n k^{-1} &= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} (k[\gamma]_d k^{-1})^n \\ &= e^{-z\Gamma}\end{aligned} \tag{2.1–10}$$

Here $[\gamma]_d$ represents the diagonal matrix having $\gamma_j$ the $j^{\text{th}}$ term in its principal diagonal and

$$\Gamma = k[\gamma]_d \, k^{-1} \tag{2.1–11}$$

Therefore we have shown that $\mu(z)$ is of the form (2.1–5) which is what we set out to do.

It is rather difficult to compute $\Gamma$ from (2.1–11) using only the above definitions of $k$ and $\gamma_j$ for one must first obtain the functions $\varphi_j(x, y)$. In dealing with the rectangular guide it is easier to use equations (1.3–5) and (1.3–13) to determine $\Gamma$.

### 2.2 *Reflection at a Single Junction*

Let a straight wave guide extending from $z = -\infty$ to $z = 0$ be joined to a curved guide of the same cross-section which extends from $z = 0$ to $z = \infty$. Let an incident wave

$$\Phi_i = \sum_{m=1}^{\infty} h_m \, e^{-z\delta_m} \, \theta_m(x, y) \tag{2.2–1}$$

come in from the left along the straight guide. The $h_m$'s are given constants, the $\delta_m$'s are the modal propagation constants for straight guides (for rectangular guides they are the $\Gamma_{mn}$'s given by (1.1–5)), and $\theta_m(x, y)$ is the $m^{\text{th}}$ eigenfunction for the straight guide (the product of a sine and a cosine for a rectangular guide).

What are the reflected and transmitted waves set up by (2.2–1)? The reflected wave is of the form

$$\Phi_r = \sum_{m=1}^{\infty} f_m^- \, e^{z\delta_m} \, \theta_m(x, y) \tag{2.2–2}$$

where the $f_m^-$'s are to be determined.

If we assume the representation

$$\Phi = \sum_{m=1}^{\infty} \mu_m(z) \theta_m(x, y) \tag{2.2–3}$$

to hold for all real values of $z$ then, since $\Phi = \Phi_i + \Phi_r$ for $z < 0$, equations (2.2–1) and (2.2–2) show that

$$\mu_m(z) = h_m e^{-z\delta_m} + f_m^- \, e^{z\delta_m}, \qquad m = 1, 2, 3, \ldots ; z < 0 \tag{2.2–4}$$

Introducing the column matrices $\mu(z)$, $h$, $f^-$ and the diagonal matrices $\exp(\pm z\Gamma_0)$ where $\Gamma_0$ is a diagonal matrix having $\delta_m$ as the $m^{\text{th}}$ term in its principal diagonal enables us to write (2.2–4) as

$$\mu(z) = e^{-z\Gamma_0} \, h + e^{z\Gamma_0} \, f^-, \qquad z < 0 \tag{2.2–5}$$

Equation (2.2–5) is more general than the expression (2.1–5) for $\mu(z)$ in that it contains waves going in both directions, but is more special in that $\Gamma_0$ is a diagonal matrix.

In the curved guide we take $\mu(z)$ to be given by (2.1–5), thus

$$\mu(z) = e^{-z\Gamma} f^+, \qquad z > 0 \tag{2.2–6}$$

where $\Gamma$ is a square matrix whose elements are assumed to be known and $f^+$ is a column matrix whose elements are to be determined along with those of $f^-$.

The conditions that the transverse components of the electric and magnetic intensities be continuous at the junction of the two guides are assumed to lead to the requirements

$$\begin{gathered}[\mu(z) \text{ in straight portion}] = [\mu(z) \text{ in curved portion}] \\ \left[\frac{d}{dz}\,\mu(z) \text{ in straight portion}\right] = \left[V\,\frac{d}{dz}\,\mu(z) \text{ in curved portion}\right]\end{gathered} \tag{2.2–7}$$

where the quantities within the brackets are evaluated at the junction and $V$ is a square matrix whose elements are constants. When the curvature of the curved portion becomes small $V$ approaches the unit matrix. For the problem at hand (2.2–7) may be written as

$$[\mu(z)]_{z=-0} = [\mu(z)]_{z=+0} \tag{2.2–8}$$

$$\left[\frac{d}{dz}\,\mu(z)\right]_{z=-0} = V\left[\frac{d}{dz}\,\mu(z)\right]_{z=+0} \tag{2.2–9}$$

in which the subscripts $z = -0$, $z = +0$ refer to the straight and curved portions, respectively, of the guide at $z = 0$.

The requirements (2.2–7) have been established for the rectangular guide in Section 1.4. Their form is also suggested by the conditions that the voltage and current be continuous at the junction of two transmission lines. Thus if we let $\mu(z)$ play the role of the voltage, the current in the first line is $-Z_1^{-1}\,d\mu(z)/dz$ and the current in the second is $-Z_2^{-1}\,d\mu(z)/dz$ where $Z_1$ and $Z_2$ denote the distributed series impedances of the two lines. It is seen that this leads to scalar equations which look like (2.2–7), but now $V$ denotes the scalar $Z_1/Z_2$ instead of a square matrix.

Setting the expressions (2.2–5) and (2.2–6) for $\mu(z)$ in the conditions (2.2–8) and (2.2–9) gives two equations which may be solved simultaneously to obtain $f^-$ and $f^+$ in terms of $h$, $\Gamma_0$, $\Gamma$ and $V$:

$$\begin{gathered} h + f^- = f^+ \\ -\Gamma_0 h + \Gamma_0 f^- = -V\Gamma f^+ \end{gathered} \tag{2.2–10}$$

$$f^- = (\Gamma_0 + V\Gamma)^{-1}(\Gamma_0 - V\Gamma)h \tag{2.2–11}$$

$$f^+ = (\Gamma_0 + V\Gamma)^{-1} 2\Gamma_0 h \tag{2.2–12}$$

Since $f^-$ and $f^+$ specify the reflected and transmitted waves, respectively, they give the answer which we are seeking.

If the curved guide should extend from $z = -\infty$ to $z = 0$ and the straight guide from $z = 0$ to $z = \infty$ the response to an incident wave $e^{-z\Gamma} h$ coming in along the curved guide would be

$$\begin{aligned} \mu(z) &= e^{-z\Gamma} h + e^{z\Gamma} f^-, \qquad z < 0 \\ \mu(z) &= e^{-z\Gamma_0} f^+, \qquad z > 0 \end{aligned} \tag{2.2-13}$$

A procedure similar to that used above shows that

$$\begin{aligned} f^- &= -(\Gamma_0 + V\Gamma)^{-1} (\Gamma_0 - V\Gamma)h \\ f^+ &= (\Gamma_0 + V\Gamma)^{-1} 2V\Gamma h \end{aligned} \tag{2.2-14}$$

where, instead of condition (2.2–9), we have used

$$V \left[ \frac{d}{dz} \mu(z) \right]_{z=-0} = \left[ \frac{d}{dz} \mu(z) \right]_{z=+0} \tag{2.2-15}$$

### 2.3 *Reflection Due to a Bend*

Let the guide be straight for $-\infty < z < -c$ and for $c < z < \infty$, and let these two portions be connected by a curved portion in which the longitudinal coordinate $z$ runs from $-c$ to $+c$. As in Section 2.2 we take the matrix propagation constants for the straight and curved portions to be the square matrices $\Gamma_0$ and $\Gamma$, respectively, and assume an incident wave, specified by the column matrix $h$, to come in from $z = -\infty$.

The column matrix $\mu(z)$ whose $m^{\text{th}}$ element appears as the coefficient of $\theta_m(x, y)$ in the representation (2.2–3) for $\Phi$ is now given by

$$\begin{aligned} \mu(z) &= e^{-z\Gamma_0} h + e^{z\Gamma_0} f^-, \qquad z < -c \\ \mu(z) &= (\cosh z\Gamma)p + (\sinh z\Gamma)q, \qquad -c < z < c \\ \mu(z) &= e^{-z\Gamma_0} f^+, \qquad c < z \end{aligned} \tag{2.3-1}$$

In these expressions $f^-$, $f^+$, $p$, $q$ are column matrices which may be determined as functions of the known matrices $\Gamma_0$, $\Gamma$, $V$ and $h$ by substituting (2.3-1) in the conditions (2.2–7) which must hold at the junctions $z = -c$ and $z = c$.

By straightforward algebra similar to that used for the analogous problem in transmission line theory we obtain

$$\begin{aligned} e^{-c\Gamma_0} f^- + e^{-c\Gamma_0} f^+ &= [-I + 2(V\Gamma \tanh c\Gamma + \Gamma_0)^{-1} \Gamma_0] e^{c\Gamma_0} h \\ e^{-c\Gamma_0} f^- - e^{-c\Gamma_0} f^+ &= [-I + 2(V\Gamma \coth c\Gamma + \Gamma_0)^{-1} \Gamma_0] e^{c\Gamma_0} h \end{aligned} \tag{2.3-2}$$

In these equations the infinite square matrix $\tanh c\Gamma$ is defined as $(\sinh c\Gamma)(\cosh c\Gamma)^{-1}$ and $\coth c\Gamma$ as its reciprocal. $\sinh c\Gamma$ and $\cosh c\Gamma$ may be defined as power series in $c\Gamma$ and may be expressed as combinations of $\exp(c\Gamma)$ and $\exp(-c\Gamma)$.

An expression for the column matrix $f^-$ may be obtained by adding the equations in (2.3–2). Before doing this it is convenient to introduce the two column matrices $x$ and $y$ defined by

$$\begin{aligned}(Vc\Gamma \tanh c\Gamma + c\Gamma_0)\, x &= c\Gamma_0\, e^{c\Gamma_0}\, h \\ (Vc\Gamma \coth c\Gamma + c\Gamma_0)\, y &= c\Gamma_0\, e^{c\Gamma_0}\, h\end{aligned} \tag{2.3–3}$$

where the scalar length $c$ has been introduced to make the various terms dimensionless. Each equation in (2.3–3) represents an infinite set of simultaneous linear equations to be solved for the elements of $x$ or $y$.

Once $x$ and $y$ are known the reflected wave is given by

$$f^- = e^{c\Gamma_0}(x + y) - e^{2c\Gamma_0}\, h \tag{2.3–4}$$

and the transmitted wave by

$$f^+ = e^{c\Gamma_0}(x - y) \tag{2.3–5}$$

# PART III

# GENTLE BENDS—GENERAL THEORY

## 3.1 *Limiting Forms Assumed for* $\Gamma$ *and* $V$

It will be shown in Part IV that for gentle circular bends in rectangular wave guides the matrix propagation constant $\Gamma$ is such that

$$\Gamma^2 = \Gamma_0^2 + F \tag{3.1–1}$$

where $\Gamma_0^2$ is the square of the matrix propagation constant for the straight guide. $\Gamma_0^2$ is a diagonal matrix having $\delta_m^2$ (which is one of the $\Gamma_{mn}^2$'s, depending on the set of modes under consideration, given by (1.1–5)) for the $m^{\text{th}}$ element in its principal diagonal. $F$ is a square matrix of infinite order in which the elements $F_{ii}$ in the principal diagonal are of order $\xi^2$ and the remaining elements $F_{ij}$, $i \neq j$ are of order $\xi$. Here $\xi = a/\rho_1$ is the ratio of the guide width to the radius of curvature of the bend. As the bend becomes more and more gentle, $\xi \to 0$.

The asymptotic expressions given in Appendix I show that, for gentle bends in rectangular guides, the square matrix $V$ which appears in the junction conditions (2.2–7) approaches a unit matrix as $\xi \to 0$. In particular $V_{ii} = 1 + v_{ii}$ where $v_{ii}$ is of order $\xi^2$, and $V_{ij}$, the element in the $i^{\text{th}}$ row and $j^{\text{th}}$ column, is of order $\xi$ when $i \neq j$.

Throughout the remainder of Part III we shall assume that $\Gamma^2$, $F$ and $V$ behave as mentioned above. In addition we assume that there is no degeneracy, i.e. all of the $\delta_m$'s are unequal to each other and to zero.

### 3.2 *Propagation in a Gentle Bend*

Here we assume that the elements of $\Gamma_0^2$ and $F$ in the expression (3.1–1) for $\Gamma^2$ are known. We wish to find the modal propagation constant $\gamma_j$ and the corresponding eigenfunction $\varphi_j(x, y)$ for the $j^{\text{th}}$ mode.

After squaring both sides of the collineatory transformation (2.1–11) connecting $\Gamma$ and the diagonal matrix $[\gamma]_d$ we obtain a relation which may be written as $k[\gamma^2]_d - \Gamma^2 k = 0$. The left hand side is a square matrix having $(\gamma_j^2 I - \Gamma^2)k_j$ as its $j^{\text{th}}$ column. Here $I$ is the unit matrix and $k_j$ is a column matrix having $k_{1j}, k_{2j}, \ldots$ as its elements ($k_j$ is the $j^{\text{th}}$ column of $k$). Thus we have a system of simultaneous linear equations in which the coefficients are furnished by the square matrix $\gamma_j^2 I - \Gamma^2$ and in which the unknowns are $k_{1j}, k_{2j}, \cdots$. Accordingly, $\gamma_j^2$ is the $j^{\text{th}}$ latent root of $\Gamma^2$ and $k_j$ is its corresponding modal column just as for the rectangular guide in Section 1.3.

In order to apply equations (A2–16) of Appendix II we set $\lambda_j = \gamma_j^2$ and $u = \Gamma^2$ so that, from (3.1–1),

$$u_{jj} = \delta_j^2 + F_{jj},\ u_{ij} = F_{ij}, \qquad i \neq j \tag{3.2–1}$$

Therefore

$$\gamma_j^2 = \delta_j^2 + F_{jj} + \sum_{s=1}^{\infty}{}' F_{js}F_{sj}/(\delta_j^2 - \delta_s^2) \tag{3.2–2}$$

$$k_{jj} = 1, \qquad k_{sj} = F_{sj}/(\delta_j^2 - \delta_s^2), \qquad s \neq j \tag{3.2–3}$$

where we have neglected terms of order $\xi^3$ in (3.2–2) and of order $\xi^2$ in $k_{sj}$, $s \neq j$. The prime on the summation indicates that the term $s = j$ is to be omitted.

When $k_{1j}, k_{2j}, \ldots$ are known the eigenfunction $\varphi_j(x, y)$ may be written as a series in $\theta_m(x, y)$ by means of equation (2.1–2).

In Section 3.3 we shall need the form assumed by the square matrix $\Gamma \tanh c\Gamma$ in a gentle bend. This matrix is used in computing the reflection from such a bend, as might be inferred from equation (2.3–3). The formula to be used is (A2–18) with $u = \Gamma^2$, $\lambda_j = \gamma_j^2$ and with the elements of the square matrix $k$ given by (3.2–3). In the diagonal matrix of (A2–18) we set

$$\begin{aligned} f(\lambda_j) &= \lambda_j^{1/2} \tanh c\lambda_j^{1/2} = \gamma_j \tanh c\gamma_j = \gamma_j t_j \\ t_j &= \tanh c\gamma_j \end{aligned} \tag{3.2–4}$$

and for the elements of $k^{-1}$ we use (A2–19) together with the line above it. When the three matrices on the right of (A2–18) are multiplied out the element $(\Gamma \tanh c\Gamma)_{ij}$ in the $i^{\text{th}}$ row and $j^{\text{th}}$ column of $\Gamma \tanh c\Gamma$ is found to be

$$(\Gamma \tanh c\Gamma)_{ij} = (\gamma_j t_j - \gamma_i t_i)k_{ij}, \qquad i \neq j \tag{3.2–5}$$

$$(\Gamma \tanh c\Gamma)_{ii} = \gamma_i t_i + \sum_m{}' (\gamma_i t_i - \gamma_m t_m)k_{im}k_{mi} \tag{3.2–6}$$

where in (3.2–5) and (3.2–6) terms of $O(\xi^2)$ ("order of") and $O(\xi^3)$, respectively, have been neglected. This is in line with the fact that the terms in the principal diagonals of our matrices must be accurate to within $O(\xi^2)$ while the remaining terms need be accurate only to within terms of $O(\xi)$. The summation with respect to $m$ runs from $m = 1$ to $\infty$ with $m = i$ omitted.

### 3.3 *Reflection from a Gentle Bend*

When the bend is gentle so that $V$ and $\Gamma$ behave according to the description given in Section 3.1, the matrix expressions for the reflection coefficients given in Section 2.3 may be evaluated. The results stated in Appendix II for "almost diagonal" matrices furnish the principal tools for this work.

It is assumed that the incident wave coming in along the straight guide from the left is $\Phi_i = \exp(-z\delta_p)\, \theta_p(x, y)$ and hence contanis only the $p^{\text{th}}$ mode. Comparing this with the general expression (2.2–1) for $\Phi_i$ shows that $h_p = 1$, $h_m = 0$, $m \neq p$, and all the elements of the column matrix $h$ are zero except the $p^{\text{th}}$ which is unity.

We start by writing the first of equations (2.3–3) as

$$(\Gamma \tanh c\Gamma + V^{-1}\Gamma_0)x = V^{-1}\, \Gamma_0\, e^{c\Gamma_0}\, h \tag{3.3–1}$$

Since $V$ approaches a unit matrix as $\xi \to 0$, the element $(V^{-1})_{ij}$ in the $i^{\text{th}}$ row and $j^{\text{th}}$ column of $V^{-1}$ is

$$\begin{aligned} (V^{-1})_{ij} &= -V_{ij}, \qquad i \neq j \\ (V^{-1})_{ii} &= 1 - v_{ii} + \sum_m{}' V_{im}\, V_{mi} \end{aligned} \tag{3.3–2}$$

where $V_{ii} = 1 + v_{ii}$, $i, j = 1, 2, 3, \ldots$ and the summation with respect to $m$ runs from 1 to $\infty$ with the term for $m = i$ omitted (as indicated by the prime on $\Sigma$). In omitting this term we are neglecting $v_{ii}^2$ because it is of order $\xi^4$. These results follow from equation (A2–2) of Appendix II. As usual, the elements in the principal diagonal are accurate to within $O(\xi^2)$ and the remaining elements to within $O(\xi)$.

It follows that $V^{-1}\, \Gamma_0\, e^{c\Gamma_0}\, h = \eta$ is a column matrix whose $i^{\text{th}}$ element is

$$\begin{aligned} \eta_i &= -V_{ip}\, \delta_p\, e^{c\delta_p}, \qquad i \neq p \\ \eta_p &= (1 - v_{pp} + \sum_m{}' V_{pm}\, V_{mp})\, \delta_p\, e^{c\delta_p}, \qquad i = p \end{aligned} \tag{3.3–3}$$

Likewise, the element in the $i^{\text{th}}$ row and $j^{\text{th}}$ column of the square matrix $V^{-1}\Gamma_0$ is $-V_{ij}\delta_j$ when $i \neq j$ and is

$$(1 - v_{ii} + \sum_m{}' V_{im} V_{mi})\,\delta_i \tag{3.3–4}$$

when $i = j$.

By combining the approximate expressions for the elements of $\Gamma \tanh c\Gamma$ (given by (3.2–5) and (3.2–6)) and $V^{-1}\Gamma_0$ we find that if $u_{ij}$ denotes the $i^{\text{th}}$ row and $j^{\text{th}}$ column of $\Gamma \tanh c\Gamma + V^{-1}\Gamma_0$ then

$$\begin{aligned} u_{ij} &= D_{ij} - V_{ij}\delta_j\,, \qquad i \neq j \\ u_{ii} &= \gamma_i t_i + \sum_m{}' D_{mi}k_{im} + \delta_i(1 - v_{ii} + \sum_m{}' C_{mi}) \\ &= \sigma_i - \delta_i v_{ii} + \sum_m{}' (D_{mi}k_{im} + \delta_i C_{mi}) \end{aligned} \tag{3.3–5}$$

In these equations we have set

$$\begin{aligned} &\sigma_i = \delta_i + \gamma_i t_i, \qquad C_{mi} = V_{im}V_{mi} \\ &D_{mi} = (\gamma_i t_i - \gamma_m t_m)k_{mi} = (\gamma_i t_i - \gamma_m t_m)F_{mi}/(\delta_i^2 - \delta_m^2) \end{aligned} \tag{3.3–6}$$

where $\gamma_i$ and $k_{mi}$ are given by (3.2–2) and (3.2–3).

We are now in a position to identify the matrix equation (3.3–1) for $x$ with the set of equations (A2–20). The quantity $\eta_i$ which appears on the right hand side of the $i^{\text{th}}$ equation in (A2–20) is given by (3.3–3). The coefficients which appear on the left hand side are the $u$'s defined by (3.3–5). Therefore, from (A2–21), when $i \neq p$,

$$\begin{aligned} x_i &= \frac{\delta_p e^{c\delta_p}}{\sigma_i\sigma_p}[-V_{ip}(\sigma_p - \delta_p) - D_{ip}] \\ &= -\frac{e^{c\delta_p}}{\delta_i(1 + t_i)(1 + t_p)}[V_{ip}\delta_p t_p + F_{ip}(\delta_i t_i - \delta_p t_p)(\delta_i^2 - \delta_p^2)^{-1}] \end{aligned} \tag{3.3–7}$$

where we have neglected higher order terms and in so doing have replaced $\gamma_j$ by the simpler $\delta_j$.

When $i = p$ (A2–21) yields

$$x_p = u_{pp}^{-1}[\eta_p + \sum_m{}' (u_{pm}u_{mp}\eta_p - u_{pm}u_{pp}\eta_m)/(u_{mm}u_{pp})] \tag{3.3–8}$$

In order to combine the second order terms in $1/u_{pp}$ with those in the rest of the expression for $x_p$ we assume that $\sigma_p$ is the major portion of $u_{pp}$. Then, approximately,

$$1/u_{pp} = \sigma_p^{-1}[1 + \delta_p v_{pp}/\sigma_p - \sum_m{}'(D_{mp}k_{pm} + \delta_p C_{mp})/\sigma_p] \tag{3.3–9}$$

This assumption, which is equivalent to assuming that tanh $c\gamma_p$ differs appreciably from $-1$, does not appear to restrict our results since tanh $c\gamma_p$ is either purely imaginary or else real and positive.

Substituting the appropriate values in (3.3–8), neglecting higher order terms, and using the definition (3.3–6) for $C_{mp}$ leads to

$$x_p = \sigma_p^{-1}\delta_p e^{c\delta_p}[1 - (1 - \delta_p/\sigma_p)v_{pp} + \sum_m{}' (\sigma_p\sigma_m)^{-1}\{C_{mp}(\sigma_p - \delta_p)(\sigma_m - \delta_m) + D_{mp}(D_{pm} - \sigma_m k_{pm}) + (\sigma_p - \delta_p)D_{pm}V_{mp} - \delta_m D_{mp}V_{pm}\}] \quad (3.3\text{–}10)$$

A reduction similar to that used in going from the first to the second line of (3.3–7) gives our final expression for $x_p$

$$x_p = \frac{\delta_p e^{c\delta_p}}{\delta_p + \gamma_p t_p} - \frac{t_p e^{c\delta_p}}{(1 + t_p)^2}\left[v_{pp} - \sum_m{}' V_{pm}V_{mp}t_m/(1 + t_m) + \sum_m{}' \frac{\delta_m t_m - \delta_p t_p}{\delta_m\delta_p(1 + t_m)(\delta_m^2 - \delta_p^2)}\{V_{pm}\delta_m F_{mp}/t_p - V_{mp}\delta_p F_{pm} + F_{pm}F_{mp}(\delta_p + \delta_m/t_p)(\delta_m^2 - \delta_p^2)^{-1}\}\right] \quad (3.3\text{–}11)$$

The above expressions for $x_i$ and $x_p$ have been derived from the first of equations (2.3–3). The second of equations (2.3–3) determines the column matrix $y$ in the same way that the first equation determines $x$ except that coth $c\Gamma$ now replaces tanh $c\Gamma$. Therefore, we may obtain expressions for the elements of $y$ by replacing the $t$'s (where $t_i$ = tanh $c\gamma_i$) by their reciprocals in the expressions for the corresponding $x$'s (i.e. in (3.3–7) and (3.3–11). The values obtained in this way lead to, when $i \neq p$,

$$\begin{aligned} f_i^- &= e^{c\delta_i}(x_i + y_i) \\ &= \delta_i^{-1} e^{c(\delta_i + \delta_p - \gamma_i - \gamma_p)} \sinh c(\gamma_i + \gamma_p)[-V_{ip}\delta_p - F_{ip}/(\delta_i + \delta_p)] \\ f_i^+ &= e^{c\delta_i}(x_i - y_i) \\ &= \delta_i^{-1} e^{c(\delta_i + \delta_p - \gamma_i - \gamma_p)} \sinh c(\gamma_i - \gamma_p)[V_{ip}\delta_p - F_{ip}/(\delta_i - \delta_p)] \end{aligned} \quad (3.3\text{–}12)$$

where we have used the expressions (2.3–4) and (2.3–5) for $f^-$ and $f^+$. When $i = p$,

$$f_p^- = e^{c\delta_p}(x_p + y_p) - e^{2c\delta_p} = -e^{2c(\delta_p - \gamma_p)}[A_1(\sinh 2c\gamma_p)/2 + A_2] \quad (3.3\text{–}13)$$

where

$$A_1 = 2v_{pp} + (\gamma_p^2 - \delta_p^2)\delta_p^{-2} - \sum_m{}' \left[V_{pm}V_{mp} + \frac{V_{pm}F_{mp} + V_{mp}F_{pm}}{\delta_m^2 - \delta_p^2}\right]$$

$$A_2 = \sum_m{}' \frac{(\cosh 2c\gamma_p - e^{-2c\gamma_m})}{2\delta_m\delta_p(\delta_m^2 - \delta_p^2)}[V_{pm}F_{mp}\delta_m^2 + V_{mp}F_{pm}\delta_p^2 + F_{pm}F_{mp}]$$

The expression for $f_p^+$ may be obtained in the same way but it is slightly more complicated.

$$f_p^+ = e^{c\delta_p}(x_p - y_p) = e^{2c(\delta_p-\gamma_p)}[1 - A_3 + A_4(\sinh 2c\gamma_p)/2 + A_5] \quad (3.3\text{–}14)$$

where

$$A_3 = (1 - e^{-4c\gamma_p})(\gamma_p - \delta_p)^2/(4\delta_p^2)$$

$$A_4 = \sum_m{}' e^{-2c\gamma_m}\left[-V_{pm}V_{mp} + \frac{V_{pm}F_{mp} - V_{mp}F_{pm}}{\delta_m^2 - \delta_p^2} + \frac{2F_{pm}F_{mp}}{(\delta_m^2 - \delta_p^2)^2}\right]$$

$$A_5 = \sum_m{}' \frac{(e^{-c\gamma_m}\cosh 2c\gamma_p - 1)}{2\delta_m\delta_p(\delta_m^2 - \delta_p^2)} \cdot \left[V_{pm}F_{mp}\delta_m^2 - V_{mp}F_{pm}\delta_p^2 + \frac{(\delta_m^2 + \delta_p^2)F_{pm}F_{mp}}{\delta_m^2 - \delta_p^2}\right]$$

There are several points we should mention about these formulas for $f_p^-$ and $f_p^+$: The summations witb respect to $m$ run from 1 to $\infty$ with the term $m = p$ omitted. $\gamma_j$ and $\delta_j$ are the propagation constants of the $j$th mode in the bend and in the straight portion, respectively. The difference $\gamma_p^2 - \delta_p^2$ may be expressed in terms of the $F$'s by equation (3.2–2). In the course of obtaining (3.3–13) and (3.3–14) relations of the following sort were used.

$$t_p(1 + t_p)^{-2} = e^{-2c\gamma_p}(\sinh 2c\gamma_p)/2$$

$$(t_m + t_p^2)(1 + t_p)^{-2}(1 + t_m)^{-1} = e^{-2c\gamma_p}(\cosh 2c\gamma_p - e^{-2c\gamma_m})$$

The term $A_3$ arises wben we subtract $(1 - t_p^2)(1 + t_p)^{-2}$ from

$$\delta_p/(\delta_p + \gamma_p t_p) - \delta_p t_p/(\gamma_p + \delta_p t_p)$$

Since $\gamma_p - \delta_p$ is $O(\xi^2)$ for a circular bend in a rectangular guide $(\gamma_p - \delta_p)^2$ is $O(\xi^4)$ and hence $A_3$ is negligible in tbe cases we shall consider.

The reflected wave set up by an incident wave of unit amplitude and containing only the $p^{\text{th}}$ mode (i.e. the incident wave described at the beginning of tbis section) is given by the column matrix $f^-$ whose elements may he obtained from (3.3–12) and (3.3–13). Likewise, the transmitted wave is given by $f^+$.

# PART IV

## GENTLE CIRCULAR BENDS IN RECTANGULAR WAVE GUIDES

### 4.1 *Propagation of Dominant Mode in a Gentle Bend—H in Plane of Bend*

When the magnetic intensity $H$ lies in the plane of the bend, $H_y = 0$, and equations (1.2–5) show that $B = 0$. Thus we have to deal only with

$A$. In order to study the dominant mode we set $\ell = 0$ in the cos $(\pi \ell y/b)$ ($A$ depends on $y$ through this factor) in the formulas of Section 1.3 wbich involve $A$ and assume the dimensions of the guide to be such that $b < a$.

We wish to determine $\gamma_1^2$, the first latent root of $\Gamma_\alpha^2$ defined by (1.3–5), from tbe approximate formula (3.2–2). In our case the elements $\delta_m^2$ of diagonal matrix $\Gamma_0^2$ are obtained by putting $n(=\ell)$ to zero in (1.1–5):

$$\delta_m^2 = \Gamma_{m0}^2 = \sigma^2 + (\pi m/a)^2, \qquad m = 1, 2, 3, \cdots \tag{4.1–1}$$

so that (3.2–2) becomes

$$\gamma_1^2 = \Gamma_{10}^2 + F_{11} - \sum_{m=2}^{\infty} F_{1m} F_{m1} a^2 \pi^{-2} (m^2 - 1)^{-1} \tag{4.1–2}$$

Tbe first task is to find the elements of the matrix $F$ where, from (3.1–1) and (1.3–5),

$$F = \Gamma_\alpha^2 - \Gamma_0^2 = (P^{-1} - I)\Gamma_0^2 + P^{-1}S \tag{4.1–3}$$

In the case under consideration $P = I + R$ where $R$ is a square matrix whose elements are very small. In fact, the asymptotic expressions leading to (A1–18) show that $R_{ii}$ and $S_{ii}$ are $O(\xi^2)$, with $\xi = a/\rho_1$, while $R_{ij}$ and $S_{ij}$ are $O(\xi)$ if $i + j$ is odd and $O(\xi^2)$ if $i + j$ is even. When the approximate value of $P^{-1}$ obtained from (A2–2) is set in (4.1–3) and the matrix multiplications carried out it is found that

$$\begin{aligned} F_{ij} &= -R_{ij}\Gamma_{j0}^2 + S_{ij} + O(\xi^2) \\ F_{ii} &= \left(-R_{ii} + \sum_{m=1}^{\infty} R_{im} R_{mi}\right)\Gamma_{i0}^2 + S_{ii} - \sum_{m=1}^{\infty} R_{im} S_{mi} + O(\xi^3) \end{aligned} \tag{4.1–4}$$

The "order of" symbol $O(\ )$ will be omitted in the following equations, it being understood that the terms in the principal diagonal are correct to within $O(\xi^2)$ and the others to within $O(\xi)$.

The values of the $F$'s which enter (4.1–2) may be computed from the asymptotic expressions (A1–18) for the $R$'s and $S$'s. They turn out to be

$$\begin{aligned} F_{1m} &= -4\xi m[4\Gamma_{10}^2 \pi^{-2}(m^2 - 1)^{-2} + 3a^{-2}(m^2 - 1)^{-1}] \\ F_{m1} &= -4\xi m[4\Gamma_{10}^2 \pi^{-2}(m^2 - 1)^{-2} + a^{-2}(m^2 - 1)^{-1}] \\ F_{11} &= \xi^2[\Gamma_{10}^2(1 - 6\pi^{-2}) + 6a^{-2}]/12 \end{aligned} \tag{4.1–5}$$

In the expressions for $F_{1m}$ and $F_{m1}$ $m$ is supposed to have the values 2, 4, 6, $\cdots$. For odd values of $m$ $F_{1m}$ and $F_{m1}$ are $O(\xi^2)$. When $i = 1$ in the expression (4.1–4) for $F_{11}$, tbe two series therein reduce to $S_3$ and $S_4$ where

$$S_p = \sum_{m=2,4,6\cdots} m^2(m^2 - 1)^{-p} \tag{4.1–6}$$

By expanding the typical terms in partial fractions and using the fact that the sums of (see, for example, page 238 of Reference[10])

$$\begin{aligned} U_q &= 1 + 3^{-q} + 5^{-q} + \cdots \\ &= (-)^{q/2-1}\tfrac{1}{2}(2^q - 1)B_q\pi^q/q! \end{aligned} \tag{4.1–7}$$

for $q = 2, 4, 6$, are $\pi^2/8$, $\pi^4/96$, $\pi^6/960$, it may be shown that

$$\begin{aligned} S_3 &= \pi^2/64, \qquad S_4 = \pi^4/768 - \pi^2/128, \\ S_5 &= (15\pi^2 - \pi^4)/3072. \end{aligned} \tag{4.1–8}$$

In (4.1–7) $B_q$ denotes the $q^{\text{th}}$ Bernoulli number. The values of $S_p$ may also be computed in succession from the two relations*

$$U_{2p} = \sum_{i=1}^{p} 2^{2i-1} C_{p+i-1,2i-1}\, S_{p+i} = \sum_{i=1}^{p+1} 2^{2i-2} C_{p+i-1,2i-2}\, S_{p+i}$$

where $C_{m,n}$ is a binomial coefficient. Still another method is to make use of the generating function

$$\sum_{p=0}^{\infty} t^p S_{p+1} = (1 + t) \sum_{m=2,4,6\cdots} (m^2 - 1 - t)^{-1} = \tfrac{1}{2} - \tfrac{1}{4}\pi x \cot \pi x$$

where $4x^2 = 1 + t$. Note that by this definition $S_1$ is $\frac{1}{2}$ in contrast to the non-convergent series obtained by putting $p = 1$ in (4.1–6).

Substituting the values for the $F$'s given by (4.1–5) in the expression (4.1–2) for $\gamma_1^2$ and using the sums (4.1–8) of the series which occur gives

$$\gamma_1^2 = \Gamma_{10}^2 - \frac{\xi^2}{4a^2}\,[1 + a^2\,\Gamma_{10}^2(1 - 6\pi^{-2}) + (a\Gamma_{10}/\pi)^4(5 - \pi^2/3)] \tag{4.1–9}$$

When the dominant mode is propagated without attenuation both $\gamma_1^2$ and $\Gamma_{10}^2$ are negative.

The general form of (4.1–9) has been obtained by both Buchholz[1] and Marshak[5] by different methods. In our notation their result is

$$\gamma_{mn}^2 = \Gamma_{mn}^2 - \frac{\xi^2}{4a^2}\left[1 + a^2\,\Gamma_{mn}^2(1 - 6\pi^{-2}m^{-2}) + \left(\frac{a\Gamma_{mn}}{\pi m}\right)^4 (5 - \pi^2 m^2/3)\right] \tag{4.1–10}$$

where $\gamma_{mn}$ is the propagation constant for the $m$, $n^{\text{th}}$ mode when the magnetic vector is in the plane of the bend.

* I am indebted to John Riordan for these relations.

### 4.2 *Reflection Due to Dominant Mode Incident upon Gentle Bend—H in Plane of Bend*

Let the system be the one described in the first paragraph of Section 2.3 and let the incident wave contain only the dominant mode. Then the matrix propagation constant is the $\Gamma_\alpha$ of Section 4.1 and the column matrix $h$ specifying the incident wave has unity for its top element and zero for its remaining elements, i.e., $p = 1$ in the formulas of Section 3.3.

We shall be interested only in the reflection coefficient, $f_{\bar{1}}$ of the dominant mode. Here we shall denote it by $g_{\bar{1}0}$, in line with the notation of equation (1.1–3), in order to distinguish it from the corresponding coefficient (which will be denoted by $d_{\bar{0}1}$) when $E$ lies in the plane of the bend.

Setting $p = 1$ in the expression (3.3–13) for the reflection coefficient and using equation (4.1–1) for $\delta_m$ gives

$$f_{\bar{1}} = g_{\bar{1}0} = -e^{2c(\Gamma_{10}-\gamma_1)}[A_1 (\sinh 2c\gamma_1)/2 + A_2] \tag{4.2–1}$$

where $\gamma_1$ has just been obtained in (4.1–9) and

$$\begin{aligned} A_1 &= 2v_{11} + (\gamma_1^2 - \Gamma_{10}^2)\Gamma_{10}^{-2} \\ &\qquad - \sum_{m=2}^{\infty} \left[ V_{1m} V_{m1} + \frac{V_{1m} F_{m1} + V_{m1} F_{1m}}{\pi^2 a^{-2}(m^2 - 1)} \right], \\ A_2 &= \sum_{m=2}^{\infty} \frac{\cosh 2c\gamma_1 - e^{-2c\gamma_m}}{2\Gamma_{m0} \Gamma_{10} \pi^2 a^{-2}(m^2 - 1)} \\ &\qquad \cdot [V_{1m} F_{m1} \Gamma_{m0}^2 + V_{m1} F_{1m} \Gamma_{10}^2 + F_{1m} F_{m1}] \end{aligned} \tag{4.2–2}$$

From (A1–18) and $V_{11} = 1 + v_{11}$ it follows that

$$\begin{aligned} v_{11} &= \xi^2(1 - 6\pi^{-2})/12 \\ V_{1m} &= V_{m1} = 8\pi^{-2}\xi m(m^2 - 1)^{-2} \end{aligned} \tag{4.2–3}$$

where $m = 2, 4, 6, \cdots$. For odd values of $m$, $V_{1m}$ and $V_{m1}$ are $O(\xi^2)$. Substituting these values together with those for the $F$'s given by (4.1–5), using the sums (4.1–8) and the expression (4.1–9) for $\gamma_1^2 - \Gamma_{10}^2$ finally leads to (after considerable cancellation)

$$A_1 = -\xi^2\Gamma_{10}^{-2}a^{-2}/4 \tag{4.2–4}$$

Likewise, for even values of $m$,

$$V_{1m}F_{m1}\Gamma_{m0}^2 + V_{m1}F_{1m}\Gamma_{10}^2 + F_{1m}F_{m1} = 16\xi^2m^2a^{-4}(m^2 - 1)^{-2} \tag{4.2–5}$$

All of the terms in the expression (4.2–1) for $g_{\bar{1}0}$ are now known (the values of $\gamma_m$ may be obtained by setting $n = 0$ in (4.1–10)). We shall make the

further approximation of putting $\Gamma_{m0}$ for $\gamma_m$. Since $\Gamma_{m0} - \gamma_m$ is $O(\xi^2)$ no serious error is introduced and we have

$$\overline{g_{10}} = \frac{\xi^2 \sinh 2c\Gamma_{10}}{8\Gamma_{10}^2 a^2} - \frac{\xi^2 8}{\pi^2} \sum_{m=2,4,6\cdots} \frac{(\cosh 2c\Gamma_{10} - e^{-2c\Gamma_{m0}})}{\Gamma_{10}\Gamma_{m0} a^2} \frac{m^2}{(m^2-1)^3} \quad (4.2\text{–}6)$$

in which

$$a\Gamma_{m0} = [\pi^2(m^2 - 1) + a^2\Gamma_{10}^2]^{\frac{1}{2}}, \qquad \xi = a/\rho_1 . \quad (4.2\text{–}7)$$

For frequencies such that only the dominant mode is propagated the ratio of the power in the reflected wave to tbe power in the incident wave is $|\overline{g_{10}}|^2$. Marshak has given an expression for this ratio which is the same as that obtained from (4.2–6) when the negligible (for his case) terms $e^{-2c\Gamma_{m0}}$ are omitted.

The corresponding expression for the transmission coefficient derived from (3.3–14) for $f_1^+$ is not as simple as (4.2–6).

### 4.3 *Propagatian of Daminant Mode in a Gentle Bend—E in Plane of Bend*

When the electric intensity $E$ lies in the plane of the bend, $E_y = 0$, and equations (1.2–5) show that $A = 0$. Here we deal with $B$ in much the same way as we dealt witb $A$ in Section 4.1. The dominant mode is obtained by setting $\ell = 1$ in the $\sin(\pi\ell y/b)$ in the formulas pertaining to $B$ in Section 1.3. It is assumed that $b > a$.

Examination of the matrices (1.3–14) indicates that, for the sake of convenience, we should call the top row of our matrices the $0^{\text{th}}$ row and the left-most column the $0^{\text{th}}$ column. In line with this we call $\gamma_0$ the propagation constant of the dominant mode in the bend. The elements $\delta_m^2$ of the diagonal matrix $\Gamma_0^2$ are obtained by putting $n(= \ell) = 1$ in (1.1–5):

$$\delta_m^2 = \Gamma_{m1}^2 = \sigma^2 + (\pi m/a)^2 + \pi^2/b^2, \qquad m = 0, 1, 2, \cdots \quad (4.3\text{–}1)$$

When we make the appropriate shift in the subscripts, equation (3.2–2) yields

$$\gamma_0^2 = \Gamma_{01}^2 + F_{00} + \sum_{m=1}^{\infty} F_{0m} F_{m0} a^2 \pi^{-2} m^{-2} \quad (4.3\text{–}2)$$

in which the elements of the matrix $F$ are to be determined from (1.3–13):

$$F = \Gamma_\beta^2 - \Gamma_0^2 = (Q^{-1} - I)\Gamma_0^2 + Q^{-1}U \quad (4.3\text{–}3)$$

As in (4.1–4) we have, with $Q = I + T$,

$$\begin{aligned} F_{ij} &= -T_{ij}\Gamma_{j1}^2 + U_{ij} \\ F_{ii} &= \left(-T_{ii} + \sum_{m=0}^{\infty} T_{im} T_{mi}\right)\Gamma_{i1}^2 + U_{ii} - \sum_{m=0}^{\infty} T_{im} U_{mi} . \end{aligned} \quad (4.3\text{–}4)$$

By using the asymptotic expressions (A1–19) for the $T$'s and $U$'s and summing the series with the value of (4.1–7) for $q = 4$ given in Section 4.1 we obtain

$$\begin{aligned} F_{0m} &= -2\xi[2\Gamma_{01}^2\pi^{-2}m^{-2} + a^{-2}] \\ F_{m0} &= -8\xi\Gamma_{01}^2\pi^{-2}m^{-2} \\ F_{00} &= \xi^2\Gamma_{01}^2/12 \end{aligned} \tag{4.3–5}$$

In these expressions $m$ is supposed to have the values 1, 3, 5, $\cdots$. When $m$ is even $F_{0m}$ and $F_{m0}$ are $O(\xi^2)$.

Substituting (4.3–5) in (4.3–2) and summing the series with the help of the values of (4.1–7) given in Section 4.1 gives

$$\gamma_0^2 = \Gamma_{01}^2 - \xi^2\Gamma_{01}^2(5 + 2a^2\Gamma_{01}^2)/60 \tag{4.3–6}$$

A result equivalent to (4.3–6) has been given by Buchholz who also gives the approximation to the propagation constant when $m > 0$ (and the electric vector in the plane of the bend). In our notation his approximation is

$$\gamma_{mn}^2 = \Gamma_{mn}^2 + \frac{\xi^2}{4a^2}\left[3 - \left(\frac{a\Gamma_{mn}}{\pi m}\right)^2 (10 + \pi^2 m^2) + \tfrac{1}{3}\left(\frac{a\Gamma_{mn}}{\pi m}\right)^4 (21 + \pi^2 m^2)\right] \tag{4.3–7}$$

In writing (4.3–7) we have corrected a misprint in Buchholz's expression. In order to agree with Buchholz's equation (5.30a) the leading term within the square brackets would have to be changed from 3 to $-3$. This change was indicated by the results obtained when our equation (3.2–2) was used to obtain special cases of (4.3–7). Probably the best way of obtaining (4.3–7) is furnished by Marshak's method (WKB approximation, out to second order terms, applied to Bessel's differential equation). If one wishes to verify (4.3–7) by using Marshak's report[5] as a guide, he should correct the misprint in Marshak's equation (12a).

### 4.4 *Reflection Due to Dominant Mode Incident upon Gentle Bend—E in Plane of Bend*

The problem here is the same as that treated in Section 4.2 except that now the electric vector lies in the plane of the bend. In line with equation (1.1–4), the reflection coefficient $f_{\bar{1}}$ for the dominant mode will be denoted by $d_{0\bar{1}}$. As in Section 4.3 the subscripts indicating the position of matrix elements will be adjusted so as to start with 0 instead of 1. The square matrix $W$ given by (1.4–5), and associated with the junction conditions for

$B$ in the same manner as $V$ is associated with $A$, now replaces $V$. Thus our expression (3.3–13) for the reflection coefficient becomes

$$f_{\overline{01}} = d_{\overline{1}} = -[A_1(\sinh 2c\Gamma_{01})/2 + A_2] \tag{4.4–1}$$

where we have neglected the difference in $\Gamma_{01}$ and $\gamma_0$ and where

$$A_1 = 2w_{00} + (\gamma_0^2 - \Gamma_{01}^2)\Gamma_{01}^{-2} - \sum_{m=1}^{\infty} \cdot \left[ W_{0m} W_{m0} + \frac{W_{0m} F_{m0} + W_{m0} F_{0m}}{\pi^2 a^{-2} m^2} \right] \tag{4.4–2}$$

$$A_2 = \sum_{m=1}^{\infty} \frac{\cosh 2c\Gamma_{01} - e^{-2c\Gamma_{m1}}}{2\Gamma_{m1}\Gamma_{01}\pi^2 a^{-2} m^2} \cdot [W_{0m} F_{m0} \Gamma_{m1}^2 + W_{m0} F_{0m} \Gamma_{01}^2 + F_{0m} F_{m0}]$$

From $W_{00} = 1 + w_{00}$ and the asymptotic expressions (A1–19) it follows that, for $m = 1, 3, 5, \cdots$

$$\begin{aligned} w_{00} &= \xi^2/12 \\ W_{0m} &= 2\xi m^{-2}\pi^{-2}, \qquad W_{m0} = 4\xi m^{-2}\pi^{-2} \end{aligned} \tag{4.4–3}$$

For even values of $m$, $W_{0m}$ and $W_{m0}$ are $O(\xi^2)$. Substitution of these values together with those for the $F$'s given by (4.3–5), using the sums (4.1–7) and expression (4.3–6) for $\gamma_0^2 - \Gamma_{01}^2$ leads to

$$A_1 = \xi^2/12$$

$$W_{0m}F_{m0}\Gamma_{m1}^2 + W_{m0}F_{0m}\Gamma_{01}^2 + F_{0m}F_{m0} = -8\xi^2\Gamma_{01}^2 a^{-2}\pi^{-2}m^{-2}$$

for $m$ odd.

Thus the reflection coefficient for the dominant mode when $E$ lies in the plane of a gentle hend of length $2c$ is approximately

$$d_{\overline{01}} = -\frac{\xi^2 \sinh 2c\Gamma_{01}}{24} + \frac{\xi^2 4\Gamma_{01}}{\pi^4} \sum_{m=1,3,5\cdots}^{\infty} \frac{\cosh 2c\Gamma_{01} - e^{-2c\Gamma_{m1}}}{m^4 \Gamma_{m1}} \tag{4.4–4}$$

where $\Gamma_{m1}$ is given by (4.3–1) and $b > a$.

## PART V

## NUMERICAL CALCULATIONS

### 5.1 *Bend in Plane of Magnetic Vector*

Let $a/b = 2.25$ and $\lambda_0/a = 1.400$ where $\lambda_0$ is the free-space wavelength of the dominant wave striking the bend. The propagation constant

$\Gamma_{10}$ of the dominant wave is obtained by setting $m = 1$, $n = 0$ in (1.1–5). The $\Gamma$'s corresponding to the higher modes may be obtained from the same formula:

$$a^2\Gamma_{10}^2 = -\left(\frac{2\pi a}{\lambda_0}\right)^2 + \pi^2 = -10.272 \qquad a\Gamma_{10} = i\,3.205$$

$$a^2\Gamma_{20}^2 = a^2\Gamma_{10}^2 + 3\pi^2 = 19.336 \qquad a\Gamma_{20} = 4.397 \qquad (5.1\text{–}1)$$

$$a^2\Gamma_{30}^2 = a^2\Gamma_{10}^2 + 8\pi^2 = 68.684 \qquad a\Gamma_{30} = 8.288$$

We shall consider a 90° bend. The approximation (4.2–6) appropriate to gentle bends becomes

$$g_{10}^- = i\xi^2[-.0122 \sin (5.03/\xi) + .0087 \cos (5.03/\xi)] \qquad (5.1\text{–}2)$$

where the exponential terms have been omitted since they are generally negligible. In (5.1–2), $\xi = a/\rho_1$ and the arguments of the sine and cosine terms arise from $2c\Gamma_{10} = \pi a\Gamma_{10}/(2\xi)$. From (4.1–9) the approximate change in the propagation constant produced by the curvature is obtainable from

$$\gamma_1^2 - \Gamma_{10}^2 = .294\xi^2/a^2 \qquad (5.1\text{–}3)$$

where $\gamma_1$ is the propagation constant of the dominant mode in the bend.

The determination of $g_{10}^{\pm}$ by matrix methods will be illustrated for a 90° bend in which $\rho_1/a = 0.6$. This makes $c/a = \rho_1\pi/(4a) = .4712$, $c\Gamma_{10} = i1.510$ and the appropriate equations in (2.3–5) and (2.3–4) become, upon setting $f_1^- = g_{10}^-$ and $f_1^+ = g_{10}^+$,

$$g_{10}^+ = e^{c\Gamma_{10}}(x_1 - y_1) = (.061 + i.998)(x_1 - y_1)$$

$$g_{10}^- = e^{c\Gamma_{10}}(x_1 + y_1) - e^{2c\Gamma_{10}} = (.061 + i.998)(x_1 + y_1) \qquad (5.1\text{–}4)$$

$$+ .993 - i.121$$

Here $x_1$, $y_1$ are the top elements in the column matrices $x$, $y$. The problem is to compute $x$ and $y$ from the matrix equations (2.3–3) with $\Gamma$ replaced by $\Gamma_\alpha$, $\Gamma_0$ defined by (1.3–2) with $\ell = 0$, and $h$ a column matrix whose elements are zero except the top one which is unity. Since the order of the matrices is infinite, an exact solution calls for an infinite amount of work. A compromise must be made between the accuracy desired and amount of work one is willing to do. The following numerical work uses third order matrices.

The first step is to compute the square matrix, obtained from (1.3–5),

$$a^2\Gamma_\alpha^2 = P^{-1}(a^2\Gamma_0^2 + a^2S) \qquad (5.1\text{–}5)$$

The elements of the diagonal matrix $a^2\Gamma_0^2$ are given by (5.1–1) and those of $P$ and $S$ by the equations and tables of Appendix I.

$$a^2\Gamma_\alpha^2 = \begin{bmatrix} 1.4429 & .8812 & .6821 \\ .8812 & 2.1250 & 1.3745 \\ .6821 & 1.3745 & 2.4879 \end{bmatrix}^{-1} \begin{bmatrix} -12.292 & 3.3447 & 1.6087 \\ -6.3039 & 16.369 & 6.9738 \\ -3.5034 & -11.3031 & 65.213 \end{bmatrix}$$

$$= \begin{bmatrix} -9.086 & -1.785 & -5.178 \\ .157 & 17.218 & -19.362 \\ .996 & -13.566 & 38.329 \end{bmatrix} \qquad (5.1\text{–}6)$$

The next step is to use (5.1–6) to evaluate the coefficients of $x$ and $y$ in (2.3–3). The square matrices $\Gamma_\alpha c \tanh \Gamma_\alpha c$ and $\Gamma_\alpha c \coth \Gamma_\alpha c$ cause most of the computational difficulties. We shall deal with these matrices by using Sylvester's theorem (an account of this theorem is given in Section 3.9 of Reference[9]). This requires the determination of the latent roots and modal rows of $a^2\Gamma_\alpha^2$. However, it is interesting to note that the matrices in question may also be computed from $c^2\Gamma_\alpha^2$ (which is easily obtained from $a^2\Gamma_\alpha^2$) by processes which employ only matrix multiplication, addition, and inversion.

Thus, setting $A^2$ for $c^2\Gamma_\alpha^2$,

$$A^{-1}\sinh A = I + \frac{A^2}{3!} + \frac{A^4}{5!} + \cdots$$

$$\cosh A = I + \frac{A^2}{2!} + \frac{A^4}{4!} + \cdots$$

$$A \coth A = (\cosh A)(A^{-1} \sinh A)^{-1}$$

$$A \tanh A = A^2(A \coth A)^{-1}.$$

Although the series always converge, they do so too slowly to be of use in our computations. The same is true of the series

$$A \tanh A = \sum_{m=1}^{\infty} 8A^2 [(2m - 1)^2 \pi^2 I + 4A^2]^{-1}.$$

For the matrices we shall encounter it appears best to use Sylvester's theorem even though this requires the determination of the latent roots and modal rows of $a^2\Gamma_\alpha^2$. The square matrix formed from the modal rows* will be denoted by $\kappa$.

* As has already been mentioned in the footnote associated with equation (1.3–9), we shall use the notation and theory set forth in Sections 3.5 and 3.6 of Reference[9].

We shall use the relations*

$$\begin{aligned} \Gamma_\alpha c \tanh \Gamma_\alpha c &= \kappa^{-1} [\gamma c \tanh \gamma c]_d \, \kappa \\ \Gamma_\alpha c \cosh \Gamma_\alpha c &= \kappa^{-1} [\gamma c \coth \gamma c]_d \, \kappa \end{aligned} \tag{5.1-7}$$

where the subscript $d$ on the brackets stands for "diagonal" matrix, the $i^{\text{th}}$ element in the principal diagonal of $[\gamma c \tanh \gamma c]_d$ being $\gamma_i c \tanh \gamma_i c$ where

$$\gamma_i c = c\lambda_i^{1/2}/a \tag{5.1-8}$$

and $\lambda_i$ is the $i$th latent root of $a^2\Gamma_\alpha^2$. In our applications $\gamma_i$ is either positive real or positive imaginary.

From (5.1-6) the $\lambda_i$'s are the roots of

$$\begin{vmatrix} \lambda + 9.086 & 1.785 & 5.178 \\ -.157 & \lambda - 17.218 & 19.362 \\ -.996 & 13.566 & \lambda - 38.329 \end{vmatrix} \tag{5.1-9}$$

$$= \lambda^3 - 46.461\,\lambda^2 - 101.96\lambda + 3464.5 = 0$$

and have the values

$$\lambda_1 = -8.886, \qquad \lambda_2 = 8.284, \qquad \lambda_3 = 47.06 \tag{5.1-10}$$

The elements $\kappa_{21}$, $\kappa_{31}$ of the modal row $[1, \kappa_{21}, \kappa_{31}]$ corresponding to $\lambda_1$ may be obtained by solving the two equations derived from the last two elements of

$$[1, \kappa_{21}, \kappa_{31}](\lambda_1 I - a^2\Gamma_\alpha^2) = 0 \tag{5.1-11}$$

namely,

$$1.785 + (\lambda_1 - 17.218)\,\kappa_{21} + 13.566\,\kappa_{31} = 0$$

$$5.178 + 19.362\,\kappa_{21} + (\lambda_1 - 38.329)\,\kappa_{31} = 0$$

When the value of $\lambda_1$ from (5.1-10) is used these equations yield

$$\kappa_{21} = .1593, \qquad \kappa_{31} = .1750$$

Likewise, the first and third elements of

$$[\kappa_{12}, 1, \kappa_{32}](\lambda_2 I - a^2\Gamma_\alpha^2) = 0$$

and the first and second elements of

$$[\kappa_{13}, \kappa_{23}, 1](\lambda_3 I - a^2\Gamma_\alpha^2) = 0$$

* This is the modal row matrix analogue of equation (11) in Section 3.6 of the Reference[9]. The modal rows of $\Gamma_\alpha$ are equal to the modal rows of $a^2\Gamma^2_\alpha$.

give

$$\kappa_{12} = .0465 \qquad \kappa_{32} = .6524$$

$$\kappa_{13} = .0165 \qquad \kappa_{23} = -.4555$$

Thus, the numbers entering (5.1–7) are

$\gamma_1 c = .4712\ (-8.886)^{1/2} = i\ 1.404,$ $\qquad \gamma_2 c = .4712\ (8.284)^{1/2} = 1.356$

$\gamma_1 c \tanh \gamma_1 c = -8.382$ $\qquad \gamma_2 c \tanh \gamma_2 c = 1.187$

$\gamma_1 c \coth \gamma_1 c = .2354$ $\qquad \gamma_2 c \coth \gamma_2 c = 1.549$

$\gamma_3 c = .4712\ (47.06)^{1/2} = 3.233$
$\gamma_3 c \tanh \gamma_3 c = 3.228$
$\gamma_3 c \coth \gamma_3 c = 3.243$

$$\kappa = \begin{bmatrix} 1 & .1593 & .1750 \\ .0465 & 1 & .6524 \\ .0165 & -.4555 & 1 \end{bmatrix}$$

For the purpose of calculation it is convenient to transform (2.3–3) by inserting (5.1–7) and premultiplying by $\kappa V^{-1}$. We obtain

$$\begin{aligned} ([\gamma c \tanh \gamma c]_d \kappa + \kappa V^{-1}\Gamma_0 c)\ x &= \kappa V^{-1} c \Gamma_0 e^{c\Gamma_0} h \\ ([\gamma c \coth \gamma c]_d \kappa + \kappa V^{-1}\Gamma_0 c)\ y &= \kappa V^{-1} c \Gamma_0 e^{c\Gamma_0} h \end{aligned} \tag{5.1–12}$$

in which

$$\kappa V^{-1} = \begin{bmatrix} 1 & .1593 & .1750 \\ .0465 & 1 & .6524 \\ .0165 & -.4555 & 1 \end{bmatrix} \begin{bmatrix} 1.1204 & .3911 & .1629 \\ .3911 & 1.2833 & .4946 \\ .1629 & .4946 & 1.3460 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} .9492 & -.1992 & .0883 \\ -.2608 & .7686 & .2339 \\ .1427 & -.7900 & 1.0160 \end{bmatrix}, \qquad \begin{aligned} \Gamma_{10} c &= i\ 1.510 \\ \Gamma_{20} c &= 2.072 \\ \Gamma_{30} c &= 3.905 \end{aligned}$$

where the elements of $V$ are obtained from the formulas and tables of Appendix I.

The $i^{\text{th}}$ equation of the set obtained by writing out the first of equations (5.1–12) is

$$\sum_{j=1}^{3} [\kappa_{ji}\gamma_i c \tanh \gamma_i c + (\kappa V^{-1})_{ij}\Gamma_{j0} c] x_j = (\kappa V^{-1})_{i1} c \Gamma_{01} e^{c\Gamma_{01}} \tag{5.1–13}$$

where $(\kappa V^{-1})_{ij}$ denotes the element in the $i^{\text{th}}$ row and $j^{\text{th}}$ column of $\kappa V^{-1}$, $\kappa_{ji}$ is the element in the $i^{\text{th}}$ row and $j^{\text{th}}$ column (note the reversal of the usual convention regarding the order of subscripts) of $\kappa$, $\kappa_{ii} = 1$, and $h$ has disappeared because it is a column matrix whose top element is unity while the remaining elements are zero. It will be noted that the only

imaginary terms in (5.1–13) occur in the coefficients of $x_0$ and arise from the imaginary quantity $\Gamma_{10}c$. By making the substitution

$$x_j = \frac{u_j c\Gamma_{10} e^{c\Gamma_{10}}}{1 + u_1 c\Gamma_{10}} \tag{5.1–14}$$

the set (5.1–13) may be reduced to

$$(\kappa_{1i}\gamma_i c \tanh \gamma_i c)u_1 + \sum_{j=2}^{3} [\kappa_{ji}\gamma_i c \tanh \gamma_i c + (\kappa V^{-1})_{ij}\Gamma_{j0} c]u_j = (\kappa V^{-1})_{i1} \tag{5.1–15}$$

in which the coefficients are all real. It should be noticed, however, that nothing is gained by making the substitution (5.1–14) when the frequency is so high that other modes in addition to the dominant are propagated.

The equation for $y$ corresponding to (5.1–15) may be obtained by replacing tanh by coth and $u$ by $v$ where now

$$y_j = \frac{v_j c\Gamma_{10} e^{c\Gamma_{10}}}{1 + v_1 c\Gamma_{10}} \tag{5.1–16}$$

Incidentally, if we set $j = 1$ in (5.1–14) and (5.1–16) and substitute in the expressions (5.1–4) for $g_{10}^{\pm}$ we may show that, since $u_1$ and $v_1$ are real,

$$| g_{10}^{+} |^2 + | g_{10}^{-} |^2 = 1 \tag{5.1–17}$$

Equation (5.1–17) may be obtained at once from the fact that the energy of the waves leaving the bend must equal the energy of the incident wave. It may also be shown that $g_{10}^{-}$ vanishes when $u_1 v_1 c^2 \Gamma_{10}^2 = 1$.

When the above numbers are set in the three equations obtained from (5.1–15) we get

$$\begin{aligned}
-8.382\, u_1 - 1.748\, u_2 - 1.122\, u_3 &= .9492 \\
.055\, u_1 + 2.780\, u_2 + 1.688\, u_3 &= -.2608 \\
.053\, u_1 - 3.105\, u_2 + 7.191\, u_3 &= .1427
\end{aligned}$$

from which

$$u_1 = -\ .0940, \qquad x_1 = .1400 + i.0113$$

The equations for $v_1$ obtained by substituting coth for tanh are

$$\begin{aligned}
.2354\, v_1 - .3732\, v_2 + .3861\, v_3 &= .9492 \\
.0717\, v_1 + 3.1417\, v_2 + 1.924\, v_3 &= -\ .2608 \\
.0534\, v_1 - 3.1143\, v_2 + 7.211\, v_3 &= .1427
\end{aligned}$$

from which

$$v_1 = 3.930, \qquad y_1 = -.1045 + i.9803$$

When these values are set in (5.1–4) we finally obtain

$$g_{10}^{+} = .9822 + i.1858$$
$$g_{10}^{-} = .0048 - i.0255$$

The following table lists values of $g_{10}^{\pm}$ obtained by the methods of this section*. Here the bend is in the plane of $H$, $a/b = 2.25$, $\lambda_0/a = 1.400$, where $\lambda_0$ is the free space wavelength of the incident wave. $\rho_1$ is the radius of curvature of the axis of the guide. The smallest possible value of $\rho_1/a$ is 0.5. The term "approx." refers to equations (5.1–2) while "1st order", "2nd order", etc. refers to the order of the matrices used in the computations. The amplitude of the reflected wave is $g_{10}^{-}$ and the amplitude of the wave sent forward is $g_{10}^{+}$ when the incident wave is of unit amplitude.

$g_{10}^{+}$

| $\rho_1/a$ | Approx. | 1st order | 2nd order | 3rd order |
|---|---|---|---|---|
| .6 | | .964 $+i$.267 | .980 $+i$.197 | .982 $+i$.186 |
| .7 | | .974 $+i$.224 | .994 $+i$.105 | .994 $+i$.111 |
| .8 | | .984 $+i$.178 | .997 $+i$.082 | .997 $+i$.082 |
| .9 | | .988 $+i$.153 | .997 $+i$.074 | .997 $+i$.073 |
| 1.0 | | .991 $+i$.135 | .998 $+i$.066 | .998 $+i$.066 |
| 1.2 | | .994 $+i$.110 | .998 $+i$.056 | .998 $+i$.056 |
| 1.5 | | .996 $+i$.084 | .999 $+i$.043 | .999 $+i$.044 |

$g_{10}^{-}$

| $\rho_1/a$ | Approx. | 1st order | 2nd order | 3rd order |
|---|---|---|---|---|
| .6 | $-i$.0280 | .0020 $-i$.0074 | .0056 $-i$.0280 | .0048 $-i$.0255 |
| .7 | $-i$.0068 | $-$.0005 $+i$.0023 | .0013 $-i$.0131 | .0007 $-i$.0066 |
| .8 | $+i$.0062 | $-$.0013 $+i$.0074 | $-$.0003 $+i$.0039 | $-$.0004 $+i$.0051 |
| .9 | $+i$.0128 | $-$.0014 $+i$.0087 | $-$.0009 $+i$.0123 | $-$.0009 $+i$.0123 |
| 1.0 | $+i$.0143 | $-$.0010 $+i$.0075 | $-$.0010 $+i$.0148 | $-$.0010 $+i$.0147 |
| 1.2 | $+i$.0079 | $-$.0002 $+i$.0018 | $-$.0005 $+i$.0086 | $-$.0005 $+i$.0085 |
| 1.5 | $-i$.0040 | $+$.0003 $-i$.0034 | $+$.0002 $-i$.0041 | $+$.0002 $-i$.0042 |

It appears that the values obtained from the first order matrices are quite far from the true values. On the other hand there is considerable agreement between the approximation and the second and third order values, especially at the larger values of $\rho_1/a$.

### 5.2 *Bend in Plane of Electric Vector*

The calculations for this case are quite similar to those presented in Section 5.1. If we are to deal with the same waveguide it is necessary to

* The computations were performed by Miss M. Darville. I am also indebted to her for the values given in the tables in Section 5.2 and Appendix I.

interchange the dimensions $a$ and $b$ so that now $b/a = 2.25$ and, for the same frequency, $\lambda_0/b = 1.400$.

For a 90° bend the approximation (4.4–4) for the reflection coefficient $d_{01}^-$ for gentle bends, i.e. for $\xi = a/\rho_1$ small, becomes

$$d_{01}^- = i\xi^2[-.0417 \sin (2.23/\xi) + .0209 \cos (2.23/\xi)]$$

where the negligible exponential terms have been neglected just as in the analogue (5.1–2) for $g_{10}^-$.

The following table, which is similar to the table at the end of Section 5.1, gives the results of computations for bends in the plane of the electric vector.

| $\rho_1/a$ | Approx. | $d_{10}^+$ 1st order | 2nd order |
|---|---|---|---|
| .6 | | $.823 + i.547$ | $.975 + i.223$ |
| .7 | | $.887 + i.447$ | $.994 + i.051$ |
| .8 | | $.921 + i.380$ | $.996 + i.042$ |
| .9 | | $.941 + i.332$ | $.997 + i.035$ |
| 1.0 | | $.954 + i.295$ | $.998 + i.031$ |
| 1.2 | | $.970 + i.242$ | $.999 + i.023$ |
| 1.5 | | $.982 + i.190$ | $1.000 + i.017$ |
| | | $d_{10}^-$ | |
| .6 | $-i.0996$ | $-.0855 + i.1284$ | $-.0020 + i.0086$ |
| .7 | $-i.0848$ | $-.0520 + i.1031$ | $+.0050 - i.0975$ |
| .8 | $-i.0706$ | $-.0330 + i.0800$ | $+.0033 - i.0792$ |
| .9 | $-i.0575$ | $-.0214 + i.0605$ | $+.0022 - i.0635$ |
| 1.0 | $-i.0457$ | $-.0137 + i.0443$ | $.0021 - i.0507$ |
| 1.2 | $-i.0258$ | $-.0051 + i.0204$ | $.0007 - i.0282$ |
| 1.5 | $-i.0051$ | $+.0001 - i.0004$ | $.0001 - i.0062$ |

The agreement between the approximation for $d_{10}^-$ and its second order matrix value is fairly good from $\rho_1/a = .7$ onward.

## APPENDIX I

### Calculation of $P_{pm}$ etc. for Circular Bend

It is convenient to write $P_{pm}$ and $Q_{pm}$ as given by (1.2–10) and (1.2–15) in the form

$$\begin{aligned} P_{pm} &= \delta_m^p + R_{pm}, \qquad p, m = 1, 2, 3, \cdots \\ Q_{km} &= \delta_m^p + T_{pm}, \qquad p, m = 0, 1, 2, \cdots \end{aligned} \tag{A1-1}$$

where $\delta_m^p$ is unity if $p = m$ and is zero otherwise and

$$R_{pm} = (2/a) \int_0^a (\rho_1^2 \rho^{-2} - 1) \sin(\pi p x/a) \sin(\pi m x/a)\, dx$$
$$T_{pm} = (\epsilon_p/a) \int_0^a (\rho_1^2 \rho^{-2} - 1) \cos(\pi p x/a) \cos(\pi m x/a)\, dx \tag{A1–2}$$

in which $\epsilon_0 = 1$, $\epsilon_p = 2$, $p = 1, 2, \cdots$

In (A1–2), (1.2–11), (1.2–16), (1.4–3), (1.4–6) we make the substitutions

$$p - m = r, \qquad u = \rho_1 \pi/a - \pi/2 = \rho_2\pi/a$$
$$p + m = s, \qquad v = \rho_1 \pi/a + \pi/2 = \rho_3\pi/a \tag{A1–3}$$
$$y = \pi x/a, \qquad w = \rho_1\pi/a, \qquad \rho = x + \rho_1 - a/2 = a(y + u)/\pi$$

Introduction of the integrals

$$I_s = (1/\pi) \int_0^\pi [w^2(y + u)^{-2} - 1] \cos sy\, dy$$
$$= (1/a) \int_0^a (\rho_1^2 \rho^{-2} - 1) \cos(\pi s x/a)\, dx$$
$$J_s = \pi \int_0^\pi \frac{\sin sy}{y + u}\, dy = \pi \int_0^a \sin(\pi s x/a)\, dx/\rho, \tag{A1–4}$$
$$K_s = \frac{\rho_1}{a} \int_0^\pi \frac{\cos sy}{y + u}\, dy$$

enables us to write

$$R_{pm} = I_r - I_s, \qquad S_{pm} = -ma^{-2}(J_s + J_r)$$
$$T_{pm} = \epsilon_p(I_r + I_s)/2, \qquad U_{pm} = m\epsilon_p a^{-2}(J_s - J_r)/2 \tag{A1–5}$$
$$V_{pm} = K_r - K_s, \qquad W_{pm} = \epsilon_p(K_r + K_s)/2$$

where $I_s$ and $K_s$ are even functions of $s$ and $J_s$ is an odd function of $s$. $\epsilon_0 = 1$ and $\epsilon_p = 2$, $p = 1, 2, 3, \cdots$. Since $w$ and $u$ depend only upon the ratio $\rho_1/a$, the values of $I_s$, $K_s$ and $J_s$ depend only upon $\rho_1/a$ and the integer $s$. These quantities are tabulated at the end of this appendix.

Setting $y + u$ equal to $t$ gives

$$J_s = \pi \int_u^v \sin s(t - u)\, dt/t$$
$$= \pi[Si(sv) - Si(su)] \cos su - \pi[Ci(sv) - Ci(su)] \sin su \tag{A1–6}$$

where $Si$ and $Ci$ denote the integral sine and cosine functions. Integrating by parts enables us to express $I_s$ in terms of $J_s$. Thus

$$\int_0^\pi (y + u)^{-2} \cos sy\, dy = u^{-1} - v^{-1} \cos s\pi - \pi^{-1} s J_s \tag{A1–7}$$

and

$$I_s = \pi^{-1}w^2[u^{-1} - (-)^s v^{-1} - \pi^{-1}sJ_s] \tag{A1–8}$$

except when $s = 0$ in which case

$$\begin{aligned} I_0 &= \pi^{-1}w^2(u^{-1} - v^{-1}) - 1 = w^2/(uv) - 1 \\ &= \pi^2/(4uv) = [(2\rho_1/a)^2 - 1]^{-1} \end{aligned} \tag{A1–9}$$

When $\rho_1/a$ is large, $u$ and $v$ are large, and the asymptotic expansion of (A1–6) gives

$$\pi^{-1}J_s \sim s^{-1}[u^{-1} - (-)^s v^{-1}] - 2!s^{-3}[u^{-3} - (-)^s v^{-3}] + \cdots \tag{A1–10}$$

When (A1–10) is placed in (A1–8)

$$I_s \sim \pi^{-1}2!w^2 s^{-2}[u^{-3} - (-)^s v^{-3}] - \cdots \tag{A1–11}$$

Formulas for $K_s$ may be obtained in much the same way.

$$\begin{aligned} K_s &= (\rho_1/a) \int_u^v \cos s(t - u)\, dt/t \\ &= (\rho_1/a)\{[Ci(sv) - Ci(su)] \cos su + [Si(sv) - Si(su)] \sin su\} \end{aligned} \tag{A1–12}$$

and when $s = 0$

$$K_0 = (\rho_1/a) \log (1 + \pi/u) \tag{A1–13}$$

The asymptotic expression is

$$a\rho_1^{-1}K_s \sim s^{-2}[u^{-2} - (-)^s v^{-2}] - 3!s^{-4}[u^{-4} - (-)^s v^{-4}] + \cdots$$

It is convenient to write the asymptotic expressions in terms of the new variable

$$\xi = a/\rho_1 \tag{A1–14}$$

When $s$ is even and greater than zero

$$J_s \sim \xi^2/s, \qquad I_s \sim 6\xi^2\pi^{-2}s^{-2}, \qquad K_s \sim 2\xi^2\pi^{-2}s^{-2} \tag{A1–15}$$

and when $s$ is odd

$$J_s \sim 2\xi/s, \qquad I_s \sim 4\xi\pi^{-2}s^{-2}, \qquad K_s \sim 2\xi\pi^{-2}s^{-2} \tag{A1–16}$$

When $s = 0$

$$I_0 \sim \xi^2/4, \qquad K_0 \sim 1 + \xi^2/12 \tag{A1–17}$$

We shall need the following asymptotic expressions which may be obtained from the above work

$$R_{11} \sim \frac{\xi^2}{4}\left(1 - \frac{6}{\pi^2}\right), \qquad S_{11} \sim -a^{-2}\xi^3/2, \qquad V_{11} = 1 + \xi^2\left(\frac{1}{12} - \frac{1}{2\pi^2}\right)$$

$$
\begin{aligned}
&R_{1m} = R_{m1} \sim 16\xi m\pi^{-2}(m^2 - 1)^{-2} \\
&S_{1m} \sim -S_{m1}, \qquad S_{1m} \sim 4a^{-2}\xi m(m^2 - 1)^{-1} \\
&V_{1m} = V_{m1} \sim 8\xi m\pi^{-2}(m^2 - 1)^{-2} \sim R_{1m}/2
\end{aligned} \tag{A1-18}
$$

where $m = 2, 4, 6, \cdots$. Also, if now $m = 1, 3, 5, \cdots$,

$$
\begin{aligned}
&T_{00} \sim \xi^2/4, \qquad U_{00} = 0, \qquad W_{00} = 1 + \xi^2/12 \\
&T_{m0} \sim 8\xi m^{-2}\pi^{-2}, \qquad U_{m0} = 0 \\
&T_{0m} \sim 4\xi m^{-2}\pi^{-2}, \qquad U_{0m} \sim 2a^{-2}\xi \\
&W_{0m} \sim 2\xi m^{-2}\pi^{-2}, \qquad W_{m0} \sim 4\xi m^{-2}\pi^{-2}
\end{aligned} \tag{A1-19}
$$

*Values of $I_s$, $J_s$ and $K_s$*

| $\rho_1/a$ | $I_0$ | $I_1$ | $I_2$ | $I_3$ | $I_4$ | $I_5$ | $I_6$ |
|---|---|---|---|---|---|---|---|
| .5 | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ |
| .6 | 2.2723 | 2.31879 | 1.82979 | 1.43755 | 1.14772 | .94432 | .78458 |
| .7 | 1.04166 | 1.28232 | .76232 | .53315 | .37692 | .28832 | .22090 |
| .8 | .64103 | .88256 | .44628 | .29206 | .18995 | .14101 | .09986 |
| .9 | .44643 | .68200 | .30111 | .18992 | .11546 | .08517 | .05908 |
| 1.0 | .33333 | .56052 | .22008 | .13637 | .07818 | .05814 | .03865 |
| 1.1 | .26041 | .47844 | .16916 | .10459 | .05772 | .04298 | .02759 |
| 1.2 | .21008 | .41905 | .13486 | .08413 | .04344 | .03361 | .02056 |
| 1.3 | .17361 | .37361 | .11059 | .06961 | .03500 | .02732 | .01633 |
| 1.4 | .14620 | .33780 | .09259 | .05926 | .03168 | .02293 | .01297 |
| 1.5 | .12500 | .30876 | .07872 | .05147 | .02315 | .01969 | .01068 |
| 2.0 | .06667 | .21803 | .04133 | .03080 | .01135 | | |
| 2.5 | .04167 | .16980 | .02564 | .02217 | .00675 | | |

| | $J_0$ | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ | $J_6$ |
|---|---|---|---|---|---|---|---|
| .5 | 0 | | | | | | |
| .6 | 0 | 3.99809 | 2.01979 | 2.31576 | 1.48355 | 1.66348 | 1.15716 |
| .7 | 0 | 3.21624 | 1.3054 | 1.58176 | .84936 | 1.04899 | .61931 |
| .8 | 0 | 2.72356 | .73339 | 1.21541 | .56683 | .77644 | .40134 |
| .9 | 0 | 2.37231 | .70698 | .99327 | .41079 | .62183 | .28546 |
| 1.0 | 0 | 2.10615 | .55663 | .84343 | .31379 | .52170 | .21578 |
| 1.1 | 0 | 1.89624 | .45091 | .73508 | .24849 | .45123 | .16981 |
| 1.2 | 0 | 1.72581 | .37335 | .65279 | .20254 | .39869 | .13768 |
| 1.3 | 0 | 1.58448 | .31450 | .58812 | .16843 | .35788 | .11413 |
| 1.4 | 0 | 1.46507 | .26878 | .53573 | .14216 | .32515 | .09636 |
| 1.5 | 0 | 1.3628 | .23251 | .49238 | .12243 | .29825 | .08254 |
| 2.0 | 0 | 1.0122 | .12817 | .35299 | .06596 | | |
| 2.5 | 0 | .8062 | .08128 | .27660 | .04140 | | |

| $\rho_1/a$ | $K_0$ | $K_1$ | $K_2$ | $K_3$ | $K_4$ | $K_5$ | $K_6$ |
|---|---|---|---|---|---|---|---|
| .5 | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| .6 | 1.43874 | .61659 | .31832 | .22547 | .15539 | .12199 | .09275 |
| .7 | 1.25423 | .42303 | .17355 | .11508 | .06971 | .05353 | .03704 |
| .8 | 1.17306 | .33141 | .11439 | .07463 | .04093 | .03195 | .02038 |
| .9 | 1.12748 | .27575 | .08261 | .05431 | .02737 | .02214 | .01325 |
| 1.0 | 1.09861 | .23761 | .06305 | .04244 | .01976 | .01675 | .00938 |
| 1.1 | 1.07891 | .20953 | .04994 | .03471 | .01509 | .01341 | .00705 |
| 1.2 | 1.06476 | .18785 | .04075 | .02935 | .01192 | .01126 | .00548 |
| 1.3 | 1.05421 | .17050 | .03391 | .02539 | .00977 | .00956 | .00443 |
| 1.4 | 1.04610 | .15626 | .02871 | .02243 | .00807 | .00838 | .00365 |
| 1.5 | 1.03972 | .14434 | .02467 | .02013 | .00682 | .00745 | .00315 |
| 2.0 | 1.02165 | .10508 | .01332 | .01330 | .00358 | | |
| 2.5 | 1.01366 | .08295 | .00838 | .01010 | .00217 | | |

## APPENDIX II

### Functions of Almost Diagonal Matrices

Let $E$ be a matrix whose elements are small in comparison with unity. It is then often possible to approximate a matrix defined as some function of the matrix $I + E$, where $I$ is the unit matrix, by the expansion

$$f(I + E) = If(1) + \frac{E}{1!}f'(1) + \frac{E^2}{2!}f''(1) + \cdots . \tag{A2–1}$$

Thus, for example, when we take $f(z)$ to be $z^{-1}$ we obtain

$$(I + E)^{-1} = I - E + E^2 - \cdots . \tag{A2–2}$$

Here we shall give similar formal results for $f(D + E)$ where now $D$ is a diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & \cdot & 0 \\ 0 & d_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & d_N \end{bmatrix} \tag{A2–3}$$

whose diagonal elements are unequal and the elements $E_{ij}$ and $E_{ji}$ are small in comparison with the absolute value of $|d_i - d_j|$. We shall restrict ourselves to a first approximation of the non-diagonal terms of $f(D + E)$ and to a second approximation of the diagonal terms. The results are closely related to the ones obtained from the perturbation theory used in wave mechanics.

We assume that $f(D + E)$ may be defined by the series

$$f(D + E) = a_0I + a_1(D + E) + a_2(D + E)^2 + \cdots \tag{A2–4}$$

where $a_n$ is a scalar and

$$(D + E)^2 = (D + E)(D + E) = D^2 + DE + ED + E^2$$

and so on. The sum of the terms independent of $E$ is $f(D)$. The terms of order $E$ are

$$\begin{aligned} &E \text{ in } D + E \\ &DE + ED \text{ in } (D + E)^2 \\ &D^2E + DED + ED^2 \text{ in } (D + E)^3 \\ &\Sigma\, D^{\ell} E D^m \text{ in } (D + E)^n \end{aligned} \tag{A2–5}$$

where the summation extends over the non-negative integer values of $\ell$ and $m$ for which $\ell + m = n - 1$. The element in the $i$th row and $j$th column of $D^{\ell} E D^m$ is $d_i^{\ell} E_{ij} d_j^m$ and hence the corresponding element in the summation in (A2–5) is

$$E_{ij} \Sigma d_i^{\ell} d_j^m = \begin{cases} (d_i^n - d_j^n)/(d_i - d_j), & i \neq j \\ n\, d_i^{n-1}, & i = j. \end{cases} \tag{A2–6}$$

Thus the terms of order $E$ in the $i$th row and $j$th column of $f(D + E)$ are, from (A2–6) and (A2–4),

$$\begin{aligned} &E_{ij} \frac{f(d_i) - f(d_j)}{d_i - d_j}, \qquad i \neq j \\ &E_{ii} f'(d_i), \qquad i = j \end{aligned} \tag{A2–7}$$

where the prime on $f$ denotes its first derivative.

The terms of order $E^2$ in $(D + E)^n$ are

$$\sum_{k,\ell,m} D^k E D^{\ell} E D^m = \sum_{k,\ell,m} [d_i^k E_{ij}]_M [d_i^{\ell} E_{ij} d_j^m]_M \tag{A2–8}$$

where the summations extend over all the non-negative integer values of $k$, $\ell$, $m$ for which $k + \ell + m = n - 2$. On the right $[d_i^k E_{ij}]_M$ denotes a square matrix whose element in the $i$th row and $j$th column is $d_i^k E_{ij}$. Likewise the second factor in brackets is a matrix having $d_i^{\ell} E_{ij} d_j^m$ in the $i$th row and $j$th column. The element in the $i$th row and $j$th column of (A2–8) is, from the rule for the product of two matrices,

$$\sum_{k,\ell,m} \sum_{s=1}^{N} (d_i^k E_{is})(d_s^{\ell} E_{sj} d_j^m) = \sum_{s=1}^{N} E_{is} E_{sj} \sum_{k,\ell,m} d_i^k d_s^{\ell} d_j^m.$$

If $i$, $s$, and $j$ are unequal the sum in $k$, $\ell$, $m$ is

$$\frac{1}{d_i - d_j} \left[ \frac{d_i^n - d_s^n}{d_i - d_s} - \frac{d_j^n - d_s^n}{d_j - d_s} \right]$$

and in case of equality the sum may he found by a limiting process. Since we are interested in this order of approximation only for the diagonal terms we set $i = j$ and obtain for the sum

$$\frac{nd_i^{n-1}}{d_i - d_s} - \frac{d_i^n - d_s^n}{(d_i - d_s)^2}, \qquad i \neq s$$

$$\frac{n(n-1)}{2!} d_i^{n-2}, \qquad i = s.$$

Thus the contribution to the $i$th diagonal element of $f(D + E)$ from terms of type (A2–8) is

$$\frac{E_{ii}^2}{2!} f''(d_i) + \sum_{s=1}^{N}{}' E_{is} E_{si} \left[ \frac{f'(d_i)}{d_i - d_s} - \frac{f(d_i) - f(d_s)}{(d_i - d_s)^2} \right] \tag{A2–9}$$

where the prime on $\Sigma$ indicates that the term $s = i$ is to be omitted.

Thus, to summarize, we may say that the first approximation to the non-diagonal term in the $i$th row and $j$th column $(i \neq j)$ of $f(D + E)$ is

$$E_{ij} \frac{f(d_i) - f(d_j)}{d_i - d_j} \tag{A2–10}$$

and the second approximation to the diagonal term in the $i$th row and $i$th column of $f(D + E)$ is

$$\begin{aligned} f(d_i) + E_{ii} f'(d) + \frac{E_{ii}^2}{2!} f''(d_i) \\ + \sum_{s=1}^{N}{}' E_{is} E_{si} \left[ \frac{f'(d_i)}{d_i - d_s} - \frac{f(d_i) - f(d_s)}{(d_i - d_s)^2} \right] \end{aligned} \tag{A2–11}$$

where the primes on $f$ denote derivatives and the prime on $\Sigma$ indicates that the term $s = i$ is to be omitted.

Two results obtained from (A2–10) and (A2–11) are of interest. For the first result we set $f(z) = z^{-1}$ and get the following approximations to the elements of $(D + E)^{-1}$:

$$\begin{aligned} &- E_{ij}(d_i d_j)^{-1}, && i \neq j \\ &d_i^{-1} - d_i^{-2} \left[ E_{ii} - \sum_{s=1}^{N} E_{is} E_{si} d_s^{-1} \right], && i = j. \end{aligned} \tag{A2–12}$$

For the second result we set $f(z) = z^{1/2}$ and ohtain the following approximations to the elements of $(D + E)^{1/2}$:

$$\begin{aligned} &E_{ij}(d_i^{1/2} + d_j^{1/2})^{-1}, && i \neq j \\ &d_i^{1/2} + \tfrac{1}{2} d_i^{-1/2} \left[ E_{ii} - \sum_{s=1}^{N} E_{is} E_{si} (d_i^{1/2} + d_s^{1/2})^{-2} \right], && i = j. \end{aligned} \tag{A2–13}$$

In (A2–12) and (A2–13) the summations include the term $s = i$.

We shall now state several results related to the above formulas. Let $u$ denote the matrix $D + E$ so that the typical element $u_{ij} = E_{ij}$, $i \neq j$, and $u_{ii} = d_i + E_{ii}$. Then the latent roots $\lambda_1, \lambda_2, \cdots \lambda_N$ of $u$ are the roots of the equation obtained by setting the determinant of $\lambda I - u$ to zero:

$$|\lambda I - u| = \begin{vmatrix} \lambda - u_{11} & -u_{12} & \cdot \\ -u_{21} & \lambda - u_{22} & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} = 0. \tag{A2–14}$$

The modal column $k_j$ corresponding to the $j$th root $\lambda_j$ satisfies the matrix equation

$$(\lambda_j I - u)k_j = 0. \tag{A2–15}$$

Since the non-diagonal elements of $u$ are small, we see from (A2–14) that we may label the roots so as to make $\lambda_j$ nearly equal to $u_{jj}$, and this together with (A2–15) shows that all the elements of $k_j$ are nearly zero except the $j$th which we may choose to be unity. When these approximate values are taken as a first approximation in the process of solving (A2–15) by successive approximations, the second approximation is found to be

$$\lambda_j = u_{jj} + \sum_{s=1}^{N}{}' \frac{u_{js} u_{sj}}{u_{jj} - u_{ss}} = u_{jj} + \sum_{s=1}^{N}{}' u_{js} k_{sj}$$

$$k_j = \begin{bmatrix} k_{1j} \\ k_{2j} \\ \cdot \\ 1 \\ \cdot \\ k_{Nj} \end{bmatrix} \qquad k_{sj} = \frac{u_{sj}}{u_{jj} - u_{ss}}, \qquad s \neq j \tag{A2–16}$$

where the 1 in the column for $k_j$ occurs as the $j$th element. This expression for $\lambda_j$ occurs in the perturbation method often used in wave mechanics.

For the modal row $\kappa_j$ corresponding to $\lambda_j$ we have in much the same way

$$\kappa_j(\lambda_j I - u) = 0$$

$$\kappa_j = [\kappa_{1j}, \kappa_{2j}, \cdots \kappa_{jj}, \cdots \kappa_{Nj}] \tag{A2–17}$$

$$\kappa_{sj} = \frac{u_{js}\kappa_{jj}}{u_{jj} - u_{ss}}$$

where the last expression is an approximation and where $\kappa_{jj}$ may be chosen at our convenience.

The results (A2–10) and (A2–11) may also be obtained from (A2–2), (A2–16) and the relation*

$$f(u) = k \begin{bmatrix} f(\lambda_1) & 0 & \cdot & 0 \\ 0 & f(\lambda_2) & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & f(\lambda_N) \end{bmatrix} k^{-1} \tag{A2–18}$$

* This is equation (12) in Section 3.6 of Reference[9]. Although proved only for polynomials it may be verified to be true for the applications which we shall make.

where $k$ is the square matrix $[k_1, k_s, \cdots k_N]$ composed of the modal columns $k_j$, and the latent roots are assumed to be distinct. The element in the $i$th row and $j$ column of $k^{-1}$ is, to our order of approximation, equal to $-k_{ij}$ when $i \neq j$ and to

$$1 + \sum_{s=1}^{N}{}' k_{js} k_{sj} \tag{A2–19}$$

when $i = j$. Here, as usual, the prime on $\Sigma$ indicates that the term $s = j$ is to be omitted.

Another result which is sometimes useful is the approximate solution of

$$\begin{aligned} u_{11}x_1 + u_{12}x_2 + \cdots + u_{1p}x_p + \cdots \quad u_{1N}x_N &= \eta_1 \\ u_{21}x_1 + u_{22}x_2 + \cdots + u_{2p}x_p + \cdots + u_{2N}x_N &= \eta_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_{p1}x_1 + u_{p2}x_2 + \cdots + u_{pp}x_p + \cdots + u_{pN}x_N &= \eta_p \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{A2–20}$$

where $|\eta_j| \ll |\eta_p|, j \neq p$, and the $u$'s are of the type assumed in (A2–14):

$$x_j = \frac{\eta_j}{u_{jj}} - \frac{u_{jp}\eta_p}{u_{jj}u_{pp}}, \qquad j \neq p \tag{A2–21}$$

$$x_p = \frac{\eta_p}{u_{pp}} + \sum_{s=1}^{N}{}' \frac{u_{ps}}{u_{ss}u_{pp}^2} (\eta_p u_{sp} - \eta_s u_{pp}).$$

The prime on $\Sigma$ indicates that the term $s = p$ (which happens to be zero here) is to be omitted. A somewhat more general result which states that the $i$th element in the column matrix $(D + E)^{-1}\eta$ is

$$\eta_i d_i^{-1} + \sum_{s=1}^{N} E_{is} d_i^{-2} d_s^{-1}(E_{si}\eta_i - d_i \eta_s) \tag{A2–22}$$

may be derived from (A2–12). Here the elements of the column matrix need not be restricted by $|\eta_j| \ll |\eta_p|, j \neq p$.

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